Slopes of energy minimizing double helices

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We study several kinds of energies that can measure geometric complexity of curves, and show some results of numerical experiments of the "best" slopes of double helices that minimize such energies.

Let us consider a double helix $\Gamma(a) = L_1(a) \cup L_2(a)$ with "*radius*" 1 and slope $a > 0$;

$$L_1(a) = \{(\cos\theta, \sin\theta, a\theta) \,|\, \theta \in \mathbb{R}\}, \;\; L_2(a) = \{(-\cos\theta, -\sin\theta, a\theta) \,|\, \theta \in \mathbb{R}\}.$$

Put $x = (1, 0, 0) \in L_1(a)$.

(1) Mutual energy based on modified potential.

Suppose $L_2(a)$ is uniformly charged. If we assume that Coulomb's force is proportional to $r^{-(\alpha+1)}$ then the "voltage" at point x is given by

$$V^{(\alpha)}_{L_2(a)}(x) = \int_{-\infty}^{\infty} \frac{\sqrt{1+a^2}\, d\theta}{\left\{(\cos\theta+1)^2 + \sin^2\theta + a^2\theta^2\right\}^{\frac{\alpha}{2}}}.$$

We assume $\alpha > 1$ in order to make the above integral converge. As the "voltage" is constant along L_1, the average "cross-term" energy of $\Gamma(a)$, $\langle E^{(\alpha)}_X(\Gamma(a))\rangle$, is proportional to the product of $V^{(\alpha)}_{L_2(a)}(x)$ and the arc-length of one turn of $L_1(a)$ which is given by $2\pi\sqrt{1+a^2}$. Therefore we can put

$$\langle E^{(\alpha)}_X(\Gamma(a))\rangle = 2\pi(1+a^2) \int_{-\infty}^{\infty} \frac{d\theta}{\left\{(\cos\theta+1)^2 + \sin^2\theta + a^2\theta^2\right\}^{\frac{\alpha}{2}}}.$$

Numerical experiments imply that $\langle E^{(\alpha)}_X(\Gamma(a))\rangle$ is a convex function of a for each $\alpha > 1$. Let $a_X(\alpha)$ be the slope that gives the minimum value of $\langle E^{(\alpha)}_X(L_1 \cup L_2)\rangle$. Then, it seems from numerical expeirments that $a_X(\alpha)$ is an increasing function of α with $a_X(1.5) \approx 1.10955, a_X(2) \approx 1.21514, a_X(2.5) \approx 1.30567, a_X(3) \approx 1.37878, a_X(10) \approx 1.65597, a_X(20) \approx 1.6988, a_X(100) \approx 1.7261$, and so on.

(2) Total energy.

Suppose $L_1(a)$ is also charged. The "voltage" at point x diverges. We can define the renormalization as follows. If α satisfies $2 \le \alpha < 3$ then we can put

$$V^{(\alpha)}_{L_1(a)}(x) = 2 \lim_{\varepsilon \to +0} \left(\int_{\varepsilon}^{\infty} \frac{\sqrt{1+a^2}\, d\theta}{\left\{(\cos\theta-1)^2 + \sin^2\theta + a^2\theta^2\right\}^{\frac{\alpha}{2}}} - \frac{1}{(\alpha-1)\left(\sqrt{1+a^2}\,\varepsilon\right)^{\alpha-1}} \right).$$

When $\alpha \ge 3$ we need additional counter terms, which are functions of α and a, for the renormalization. Now we can define the average energy by $\langle E^{(\alpha)}(\Gamma(a))\rangle = 2\pi\sqrt{1+a^2}\left(V^{(\alpha)}_{L_2(a)}(x) + V^{(\alpha)}_{L_1(a)}(x)\right)$.

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