Analytical solution to a fracture problem in a tough layered structure

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Nacre causes the shining beauty of pearl due to its remarkable layered structure, which is also strong. We reconsider a simplified layered model of nacre proposed previously [Okumura and de Gennes, Eur. Phys. J. E 4, 121 (2001)] and obtain an analytical solution to a fundamental crack problem. The result asserts that the fracture toughness is enhanced due to a large displacement around the crack tip (even if the crack-tip stress is not reduced). The derivation offers ideas for solving a number of boundary problems for partial differential equations important in many fields.

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Nacre is not only beautiful but also strong. A prominent feature of this substance is a layered structure at submicrometer scale. This structure is the origin of the shining beauty of pearl, and it is known to possess a remarkable toughness [1,2] which has been studied over many years [3–14].

One important factor governing the strength of a material is the stress concentration around ubiquitous small cracks in materials: around a crack tip the stress is enhanced, which breaks the bonds to initiate failure. One can feel this on the macroscopic level, just with a piece of paper: the sheet is actually rather strong if one tries to break it by applying tensile force with the hands, but it easily breaks if one introduces a cut (i.e., crack) by a sharp knife in the middle in the direction perpendicular to the tensile direction.

Among many ideas on the toughening mechanism of nacre, the possibility of reduction of such stress enhancement around the tip is suggested [5] by use of a simplified elastic model mimicking the structure and by consideration of a semi-infinite crack in the middle of an infinitely long plate of width 2L [Fig. 1(a)], which allows an analytical solution.

In the model, hard layers (thickness $d_h$ and Young modulus $E_h$) are glued together by soft layers (thicknesses $d_s$ and Young modulus $E_s$) as in Fig. 1 where the period of the stripe, $d$, is defined as $d=d_s+d_h$. We introduce small parameters $e_E=d/d_s$ and $e_d=d_h/d_s$:

$$E_s = e_E E_h, \quad d_s = e_d d_h.$$  (1)

We require the following properties, which are important characteristics of our model:

$$e = e_E d/d_s \equiv e_E/e_d \ll 1. \quad (2)$$

Nacre may be modeled typically by $e_E \equiv 1/5000$ and $e_d \equiv 1/100$, where $e \equiv 1/50$.

In our previous paper [5], we showed that, for a line crack running in the $x$ direction under the plane strain condition (thick plate), as in Figs. 1(a) and 1(b), the elastic energy per unit volume of the model can be reduced to

$$f = \frac{E_h}{2(1-\nu^2)} \left( \frac{\partial u_x}{\partial y} \right)^2 + \frac{E_0}{4(1+\nu)} \left( \frac{\partial u_y}{\partial x} \right)^2$$  (3)

with the formal relations $\sigma_{ij} \sim \sqrt{e} \delta_{ij}$ and $u_{ij} \sim \sqrt{e} u_{ij}$, where $u_{ij}$ and $\nu$ are the displacement field and Poisson’s ratio, respectively.

At equilibrium, by minimizing the volume integral over $x$ and $y$ of Eq. (3) with respect to $u_{ij}$, we obtain the following reduced Laplace equation for the dominant displacement field $u_x$:

$$\left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) u_x = 0,$$  (4)

where a reduced $x$ coordinate has been introduced:

$$\tilde{x} = x/\sqrt{e} \quad \text{with} \quad e = e(1-\nu)/2.$$  (5)

As announced, we consider a finite line crack of length $2a$ in Fig. 1(b), for which the following boundary conditions are appropriate in the upper half plane ($y>0$):

$$u_x = \begin{cases} u_0 & \text{at } y = L, \\ 0 & \text{for } y = 0, \; x < -a \text{ or } x > a, \end{cases}$$

$$\frac{\partial u_x}{\partial y} = 0 \quad \text{for } y = 0, \; -a < x < a.$$  (6)

Note that the original field $u_x$ on the whole $xy$ plane has a discontinuous jump at the crack surface ($-a<x<a, \; y=0$): $u_x$ is positive for $y=0^+$ but negative for $y=0^-$. To overcome the discontinuous jump at the crack surface, we consider an auxiliary field on the whole $xy$ plane defined as
The desired analytic function 

\[ u = \begin{cases} u_y, & y > 0, \\ -u_y, & y < 0. \end{cases} \]  

(7)

By introducing this auxiliary field \( u \), which is always positive and has no discontinuous jump at the crack surface, the original Cauchy boundary problem [both \( u_y \) and its derivative appear in boundary conditions as in Eq. (6)] is changed into a simpler Dirichlet boundary problem (\( u_y \) is always given at the boundary):

\[ u = \begin{cases} u_0 & \text{at } y = \pm L, \\ 0 & \text{for } y = 0, \pm a \leq x \leq a. \end{cases} \]  

(8)

A solution of the Laplace equation on the \( (\hat{x}, y) \) plane, \( u \), satisfying Eq. (8), can be obtained by finding the conformal mapping from \( z = \hat{x} + iy \) to \( w = f(z) \) where \( \text{Im } w \) meets requirements specified by Eq. (8); we obtain

\[ u_y = \text{Im } w \quad \text{for } y > 0, \]  

(9)

because \( \text{Im } w \) of an analytic function \( w \) satisfies the Laplace equation. This implies, for \( y > 0 \),

\[ \sigma_{yy} = E \frac{d^2 w}{dz^2} \quad \text{with } E = \frac{E_h}{1 - \nu^2}. \]  

(10)

The desired analytic function \( w = f(z) \) can be found by introducing another complex plane \( \xi \) in order to consider a quasi-Schwarz-Christoffel transformation from \( \xi = \hat{x} + i \eta \) to \( z = \hat{x} + iy \) where

\[ \frac{dz}{d \xi} = k \frac{\xi^2 - 1}{(\xi + m)(\xi - p)(\xi - q)}, \]  

with positive real numbers \( m, p, \) and \( q \).

<table>
<thead>
<tr>
<th>( \xi )</th>
<th>( 0 &lt; \xi &lt; p )</th>
<th>( p &lt; \xi &lt; q )</th>
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<td>( \text{arg} \left( \frac{\xi}{(\xi - p)(q - \xi)} \right) )</td>
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<td>( -\pi )</td>
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<tr>
<td>( 2 \text{ arg} \left( \frac{\xi - p}{\xi} \right) )</td>
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<td>( 2\pi )</td>
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FIG. 2. Paths from \( A \) to \( G \) on the \( z \) (a) and \( w \) (c) planes corresponding to the path from \( A \) to \( G \) on the \( \zeta \) plane (b). Real parts of the points \( D, D', F, F' \) in (a) and \( C, C', F, F' \) in (c) are \( \infty \) and those of \( A, C, C' \) in (a) and \( A, D, D', G \) in (c) are \( -\infty \).

This transformation is examined in some detail to make the subsequent arguments understandable. Let us move \( \zeta \) from \( -\infty + i0^+ \) to \( +\infty + i0^+ \) on the real axis of \( \zeta \) (the upper plane) passing through the intermediate points \( \zeta = -1, \) \( -m, p, 1, q \) [see Fig. 2(b)] and define the range of the principal value of the argument \( \text{arg}(Z) \) of a complex variable \( Z \) to be \( -\pi < \text{arg}(Z) \leq \pi \): if \( Z \) is on the real axis, with \( \text{Re}[Z] \) positive, \( \text{arg}(Z) = 0 \), but, with \( \text{Re}[Z] \) negative, \( \text{arg}(Z) = \pi \) and \( -\pi \) when \( \text{Im}[Z] \) is \( 0^+ \) and \( -0^+ \), respectively. Note that under this convention, the relation \( \log(1/Z) = -\log Z \) is correct, but, in general, the relation

\[ \log(z_1 z_2) = \log(z_1) + \log(z_2) \]  

(12)

is correct only when

\[ -\pi < \text{arg}(z_1), \text{arg}(z_2), \text{arg}(z_1 z_2) \leq \pi. \]  

(13)

Note that “log” stands for the natural logarithmic function. If the sum \( \text{arg}(z_1) + \text{arg}(z_2) \) is not in the above range, we have to add on the right-hand side of Eq. (12) \( 2\pi i \) or \( -2\pi i \) so that the imaginary part of both sides of Eq. (12) becomes equal. For example, we obtain

\[ \log(-z_1) = \begin{cases} i\pi + \log(z_1), & -\pi < \text{arg}(z_1) \leq 0, \\ -i\pi + \log(z_1), & 0 < \text{arg}(z_1) \leq \pi. \end{cases} \]  

(14)
Let us go back to the above movement on the real axis in the ζ plane in Fig. 2(b), from A to G. During this movement, for an increase in ζ, Re[dζ] is positive and Im[dζ]=0, i.e., arg(dζ)=0, so that

\[ \arg(dz) = \arg(dz) - \arg(d\xi) = \arg(dz/d\xi) \]  
\[ = \arg(k) + \arg(\xi + 1) - \cdots - \arg(\xi - q). \]  

The first equality holds because of arg(dζ)=0 and the third follows from Eq. (11). From Eq. (16), we see that, for example, when we pass ζ=q from the left to the right, arg(ζ-q) changes from π to 0, with the other terms in the last line in Eq. (16) unchanged; arg(dz) is constant when ζ approaches q from the left, is changed by π at ζ=q, and is constant again when ζ goes away from q; z could move on a straight line (ζ<q) and jump to another straight line (ζ>q) with a rotation by the angle π at ζ=q, with ζ following the path E→F→G in Fig. 2(a). Indeed, we can choose unknowns (k,m,p,q plus an integration constant k') in such a way that the movement of ζ (with Im ζ=0), A→B→C→D→E→F→G in Fig. 2(b) is mapped to the movement A→B→C→D→E→F→G in Fig. 2(a) on the z plane. For this purpose, we integrate Eq. (11) and determine the unknowns by the following conditions: (I) z has to make specified jumps at passages C→C', D→D', and F→F' (e.g., at the passage F→F', z jumps from ∞ to x+iL) and (II) z should be ±a when ζ = ±1 where a=ai/√ε. From I and II, we find

\[ z = \frac{L}{\pi} \log \frac{(q-p)\xi}{(\xi-p)(q-\xi)} \]  

with

\[ m = 0, \quad p = \tanh \frac{\pi a}{2L} < 1, \quad q = \coth \frac{\pi a}{2L} > 1. \]  

Indeed, we can easily confirm that the transformation given in Eq. (17) satisfies the above required conditions. From Table I, for example, at the passages C→C' (ζ=0) and D→D' (ζ=p),

\[ \text{Im } z = \frac{L}{\pi} \arg \left( \frac{\xi}{(\xi-p)(q-\xi)} \right) \]  

jumps by the amount -L and L, respectively. At ζ=±1, the right-hand side of Eq. (17) reduces to ±a by noting the relation

\[ \frac{1}{q-p} = \cosh \frac{\pi a}{2L} \sinh \frac{\pi a}{2L} \]  

Next we find a transformation ζ→w (appropriate for the desired transformation z→w), not in the form of the Schwarz-Christoffel transformation, by noting that, when ζ is on the real axis (with Im ζ=0), Im W=ui for ζ<ξ0 while Im W=ui2 for ζ>ξ0, in a transformation ζ→w,

\[ W(\zeta_0,u_1,u_2) = iu_2 + \frac{u_1 - u_2}{\pi} \log(\xi - \xi_0). \]  

The boundary condition in Eq. (8) states that, along the paths A→B→C and D→E→F in Fig. 2(a), u=Im w should be zero while this should be u0 on the paths C'→D and F'→G. This results in the following condition for the transformation ζ→w (m=0), from Fig. 2(b) on the real axis of ζ (with Im ζ=0), Im w=0 for ζ<−w and p<ζ<q while Im w=u0 for −m<ζ<p and q<ζ. This condition is satisfied for

\[ w = W(0,0,u_0) + W(p,0,-u_0) + W(q,0,u_0). \]  

Indeed we can check this by Table II.

<table>
<thead>
<tr>
<th>ζ&lt;0</th>
<th>0&lt;ζ&lt;p</th>
<th>p&lt;ζ&lt;q</th>
<th>q&lt;ζ</th>
</tr>
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<tbody>
<tr>
<td>Im W (0,0,u0)</td>
<td>0</td>
<td>u0</td>
<td>u0</td>
</tr>
<tr>
<td>Im W (p,0,-u0)</td>
<td>0</td>
<td>0</td>
<td>−u0</td>
</tr>
<tr>
<td>Im W (q,0,u0)</td>
<td>0</td>
<td>0</td>
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Equation (22) can be manipulated with the aid of Eq. (12):

\[ w = \frac{u_0}{L} \log \frac{\xi - p}{\xi - q} + 2 \arg \left( \frac{\xi - p}{\xi - q} \right) \]  

Here, ζ is given by Eq. (17). As a matter of fact, we can confirm that w given in Eq. (23) as a function of ζ moves on the path A to G in Fig. 2(c) as ζ goes along the path A→G in Fig. 2(b): we can check that Im w=0 at ζ=1+i0* and that w makes necessary jumps at intermediate points [e.g., at the passage C to C' (ζ=0), Im w jumps by the amount iu0, with the aid of Table I, by noting the relation

\[ \text{Im } w = \frac{u_0}{\pi} \arg \left( \frac{\xi}{(\xi-p)(q-\xi)} \right) + 2 \arg \left( \frac{\xi-p}{\xi-q} \right) \]  

obtained from Eqs. (19) and (23). Thus, Eq. (23) guarantees the movement on the w plane from A to G along straight lines with rotations by the angle π at C, D, and G [Fig. 2(c)] when ζ moves on the real axis from A to G [Fig. 2(b)], as can be checked by using a relation similar to Eq. (16). We stress here that the transformation in Eq. (23) is not the Schwarz-Christoffel transformation: the derivative of Eq. (23),

\[ \frac{dw}{d\xi} = \frac{(\xi - \xi_0)(\xi - \xi_2)}{(\xi - p)(\xi - q)}, \]  

with points ξ±=p ± i1−p' located off the real axis.

Finally, we derive the transformation z→w. From Eq. (17), we obtain

\[ \xi = \phi(z) + \varphi(z) \]  

[the reason we have selected the plus sign in front of ϕ(z) can be seen in Eq. (36) below], where

\[ 2\phi(z) = (p+q) - (q-p)e^{-\pi/L}, \]  

\[ \varphi(z) = \left[ \phi(z)^2 - 1 \right]^{1/2}. \]  

Equation (23) with Eqs. (26)–(28) completely defines the desired transformation z→w.
From this Eq. (23), the displacement field and stress field are analytically obtained through Eqs. (9) and (10):

\[ u_y = \frac{u_0}{L} y + \frac{2u_0}{\pi} \arctan \left( \frac{\text{Im} \left[ 1 - \frac{p}{\phi(z) + \varphi(z)} \right]}{\text{Re} \left[ 1 - \frac{p}{\phi(z) + \varphi(z)} \right]} \right), \]  
(29)

\[ \sigma_{yy} = \sigma_0 \left[ 1 + \text{Re} \left( \frac{p(p + q - 2)\varphi(z)}{\varphi(z)(\phi(z) - p + \varphi(z))} \right) \right], \]  
(30)

where \( \sigma_0 \) is a measure of the remote stress,

\[ \sigma_0 = E u_0 / L. \]  
(31)

The crack shape and stress concentration around the crack tip are examined by putting \( z = \hat{a} - \hat{r} + i0^+ \) in Eq. (29) and \( z = \hat{a} + \hat{r} + i0^+ \) in Eq. (30), where \( \hat{r} = \sqrt{\hat{r}} / \sqrt{\epsilon} \), so that \( \hat{r} \) represents the distance from the crack tip. For small \( \Delta \), we have

\[ \phi(\hat{a} + \Delta) = 1 + \frac{\pi(p + q - 2)}{2L} \Delta, \]  
(32)

\[ \varphi(\hat{a} + \Delta) = \left[ \pi(p + q - 2) \Delta / L \right]^{1/2}, \]  
(33)

where

\[ p + q - 2 = (q - p)e^{-\pi\hat{r}/L} > 0. \]  
(34)

Thus, at \( z = \hat{a} - \hat{r} + i0^+ \), we have

\[ \frac{1}{\phi(z) + \varphi(z)} = \frac{1}{1 - \left[ \pi(p + q - 2)\sqrt{\hat{r}/L} \right]^{1/2}} \]  
(35)

and

\[ u_y = \frac{2u_0 p \left[ \pi(p + q - 2)/L \right]^{1/2}}{1 - p} = 2\kappa u_0 \sqrt{\frac{\hat{r}}{\pi L}} \]  
(36)

[the wrong sign in Eq. (26) would give the wrong sign for this expression]. At \( z = \hat{a} + \hat{r} + i0^+ \), we have

\[ \sigma_{yy} = \sigma_0 \text{Re} \left( \frac{p(p + q - 2)}{\varphi(z)(1 - p)} \right) = \kappa \sigma_0 \sqrt{\frac{L}{\pi p}}, \]  
(37)

where

\[ \kappa = \frac{p\sqrt{p + q - 2}}{1 - p} = \frac{e^{\pi\hat{r}/L} - 1}{\sqrt{e^{2\pi\hat{r}/L} - 1}} \]  
(38)

We consider the limit \( L \ll \hat{a} \), which includes the case \( L = a \), for which we have \( \kappa = 1 \). In this limit, for \( y = 0^+ \), the displacement at \( x = a - r \) and the stress at \( x = a + r \) are given by

\[ u_y = 2e^{1/4}K_L \sqrt{\hat{r}} / E, \quad \sigma_{yy} = e^{1/4}K_L \sqrt{\hat{r}} / \sqrt{\epsilon} \]  
(39)

The tip displacement is enhanced by a factor \( e^{1/4} \) while the tip stress concentration is reduced by a factor \( e^{1/4} \), compared with a monolithic material of the same size with the same crack.

We next consider the limit \( L \gg \hat{a} \), corresponding to an infinite plate as in the Griffith problem, where \( \kappa = \sqrt{\pi \hat{a}} / 2L \). In this limit, for \( y = 0^+ \), the displacement at \( x = a - r \) and the stress at \( x = a + r \) are given by

\[ u_y = e^{-1/2} \frac{2K_a \sqrt{\hat{r}}}{E}, \quad \sigma_{yy} = \frac{K_a}{\sqrt{\epsilon}} \]  
(40)

Tip displacement is enhanced by a larger factor \( e^{-1/2} \) while the tip-stress singularity is the same as that of the pure material of the same size with the same crack [see Eq. (39)].

As a matter of fact, most of the scaling structures in expressions obtained from the present analytical solution can be reproducible from simple scaling arguments, which require a separate presentation [15], where we show a useful general formula on the fracture toughness. Here, the fracture toughness is defined in a standard way [16] as the value of the energy release rate, \( G \), at the critical of failure where \( G \) is given by \( G = -d\Pi / d\hat{a} \) and \( \Pi \) is the elastic potential energy per unit width of the crack front. In the general formula, the fracture toughness is enhanced by

\[ \lambda = (\lambda_n / \lambda_0) (d / a_0), \]  
(41)

where \( \lambda_n \), \( \lambda_0 \), and \( a_0 \) are the tip-displacement enhancement factor, the tip-stress reduction factor, and the typical size of Griffith flaws in the hard sheets: the tip-displacement enhancement \( (\lambda_n \gg 1) \) and the tip-stress reduction \( (\lambda_0 \ll 1) \) are two independent origins of the fracture toughness. In the above two different limits of \( L \ll \hat{a} \) and \( L \gg \hat{a} \), we obtained different sets of factors \( (\lambda_n, \lambda_0) = (e^{-1/4}, e^{1/4}) \) and \( (e^{-1/2}, 1) \), respectively [see Eqs. (39) and (40)], so that the physical origins of the toughening are different: in the limit \( L \ll \hat{a} \), similar to the result in [5], both enhancement of tip displacement and reduction of tip-stress concentration contribute, while only a stronger tip-displacement enhancement plays a role in the limit \( L \gg \hat{a} \). However, the enhancement factor of the fracture toughness \( \lambda \) given in Eq. (41) is the same in the two different limits of \( L \ll \hat{a} \) and \( L \gg \hat{a} \): \( \lambda = e^{-1/2} (d / a_0) \). From this robust universal relation, we could deduce the reason of ubiquitance of strong nanostructured soft-hard composites in nature [15].

The physical reasons for the above deformation enhancement and reduction of the stress concentration due to the layered structure can be understood (with the help of the physical pictures obtained from the scaling arguments) [15]: for example, in a discrete model where soft and hard layers are treated as blocks (distinguishing stress and strain in two different soft and hard phases on a scale smaller than \( d \)), displacements in soft layers are significant (this is partially confirmed in simulation [17]), which contributes to the displacement enhancement in the present continuum model applicable on scales larger than the block sizes. In conclusion, we have analytically solved the finite-crack problem for a layered structure mimicking nacre, i.e., a boundary problem for the Laplace equation on a reduced plane \( (\hat{x}, \hat{y}) \). This was carried out by replacing a Cauchy problem for a singular field \( u_r \), with a Dirichlet problem for a regular auxiliary field \( u \) and by finding an appropriate conformal mapping \( z \mapsto \hat{z} \).
via another complex plane $\zeta$ where a non-Schwarz-Christoffel transformation was involved. Technically, the derivation will help in solving certain boundary problems in many different fields. Physically, the result clarifies two origins of strength of the special layered structure of macroscopic size: (i) a reduced crack-tip stress and (ii) a strong displacement around the crack tip. In particular, even without (i), due to (ii) alone, the fracture toughness can be increased significantly.

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