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Scaling Relation in Fracture of the Materials with Elastoplastic Response Inaccessible by Scaling Laws

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Strong materials and a solid method of estimating their toughness are important for manufacturing materials valuable for human life, such as rubber and plastic. Such strong materials often exhibit a complex response to external force, which resists description by simple scaling laws. In addition, even for simple nonlinear materials, no theoretical or experimental reports have been available on a clear scaling law between fracture stress and crack size, which would, if available, provide a solid test for toughness. Here, we perform experiments on thin sheet samples to make important length scales well-separated. This practically suppresses all the finite-size effects so that we succeed in finding a clear scaling law for fracture (that between failure stress and crack size) by studying nonlinear polymer sheets. This leads to plausible estimates of the fracture toughness. Remarkably, we experimentally find the scaling law even though the nonlinearity in force response is not so simple as described by power laws. The fracture scaling can be explained by a theory developed here for simple nonlinear materials and we expect that this theory will be valid for many other materials with a complex nonlinearity, as demonstrated here. This clarifies the advantage of testing thin sheets. The scaling law established here can be regarded as a nonlinear extension of the Griffith's formula and holds also for thick samples: the nonlinear Griffith's formula is also applicable to three-dimensional bulk objects.

KEYWORDS: fracture mechanics, fatigue and cracks, systems obeying scaling laws, physical properties of polymers

1. Introduction

Fracture mechanics aims to predict conditions under which materials fail. It is extremely important in various fields, not only in physical sciences but also in more applied fields such as the manufacture of rubber and plastic products.[1)] For example, one of the key concepts, i.e., stress concentration, is frequently indispensable for understanding the toughness of natural composites.[2–4)] Nonlinearity in fracture is a ubiquitous and important issue, for example, as in studies on crack patterns,[5–10)] on rapid,[11–14)] slow[15)] and delayed fractures,[16,17)] and on the fracture of various types of heterogeneous materials.[18–21)]

In this paper, we revisit a static aspect of nonlinearity in fracture. This issue was explored to result in theoretical schemes such as the HRR theory (developed by Hutchinson, Rice, and Rosengren[22,23)]), J-integral, and crack-tip opening displacement.[1,24)] However, experimentally, simple scaling laws between failure stress and crack size have never been discussed for real complex materials. Accordingly, the experimental determination of fracture surface energy, a measure of fracture toughness, of nonlinear materials is highly non-trivial in practice.[1)] Here, we exclude from our experiments finite-size effects by making the characteristic length scales, i.e., size and thickness of samples and crack sizes, well-separated to simplify the problem. As a result, we find a clear and simple scaling law for the stress at the onset of crack propagation in real nonlinear plastic materials, remarkably, even though the materials are nontrivially nonlinear, namely, the stress–strain relation cannot be expressed as clear power laws. We demonstrate and justify that we can treat such practical and complex nonlinear plastic materials with a simple and generalized Griffith's fracture criterion. As a result, we provide direct access to the fracture surface energy of nonlinear materials. This could lead to a new fundamental tool for developing strong materials.

2. Experiment

To study the fracture of materials in a simplified situation under a plane stress condition, we developed a simple experimental setup.[25)] This setup allows us to determine the stress–strain curves of a thin-sheet sample of about $50 \times 50\,\mathrm{cm}^2$ area and failure stresses of such a sample with macroscopic line cracks of millimeter to centimeter sizes. We can clamp a sheet in between a fixed pair of plates (bottom) and another movable pair of plates (top), while the latter pair is moved upwards via a wire (and a force gauge) by a motor at a fixed speed of about 0.1 mm/s. By simultaneously measuring tensile forces via a force gauge and extensions of a sheet without cracks, we obtain a stress–strain curve. Failure stresses are measured by introducing a sharp line crack of length ($2a$) from 3 mm to 5 cm at the center with a knife and by measuring the force when the initial crack starts to expand. Note that the sample size is well separated from the crack size and both are separated from the sample thickness (approximately 0.01 mm). We tested two types of polyethylene sheet cut out from a commercial plastic bag: (A) one with a thickness of 0.03 mm (Marubeni Plax) and (B) another with a thickness of 0.015 mm (Fukusuke Kogyo). This setting practically suppresses finite-size effects.

3. Results

The experimental stress–strain curves for the samples are given in Fig. 1. As shown in the insets (log–log plots), the curves approach a power law (i.e., a straight line) in a large-strain region. The range is, however, relatively narrow or the scaling is not clear (especially in sample B). Nonetheless, if we plot failure the stress σ_f as a function of the (half) crack length a in Fig. 2, the curve exhibits a clear scaling law well over one order of magnitude in terms

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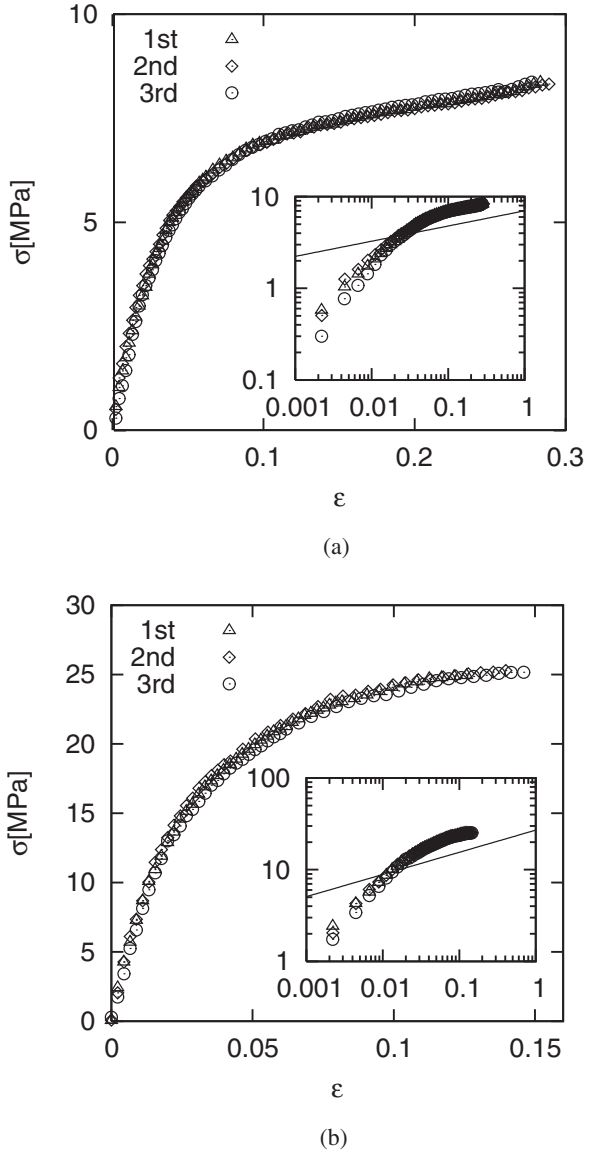

Fig. 1. Stress–strain relations obtained from sheet samples, A (a) and B (b). The results of three separate measurements (first, second, and third) are shown to clarify the good reproducibility. The solid lines in the inset stand for the slope $1/n_{ex}$, as discussed below.

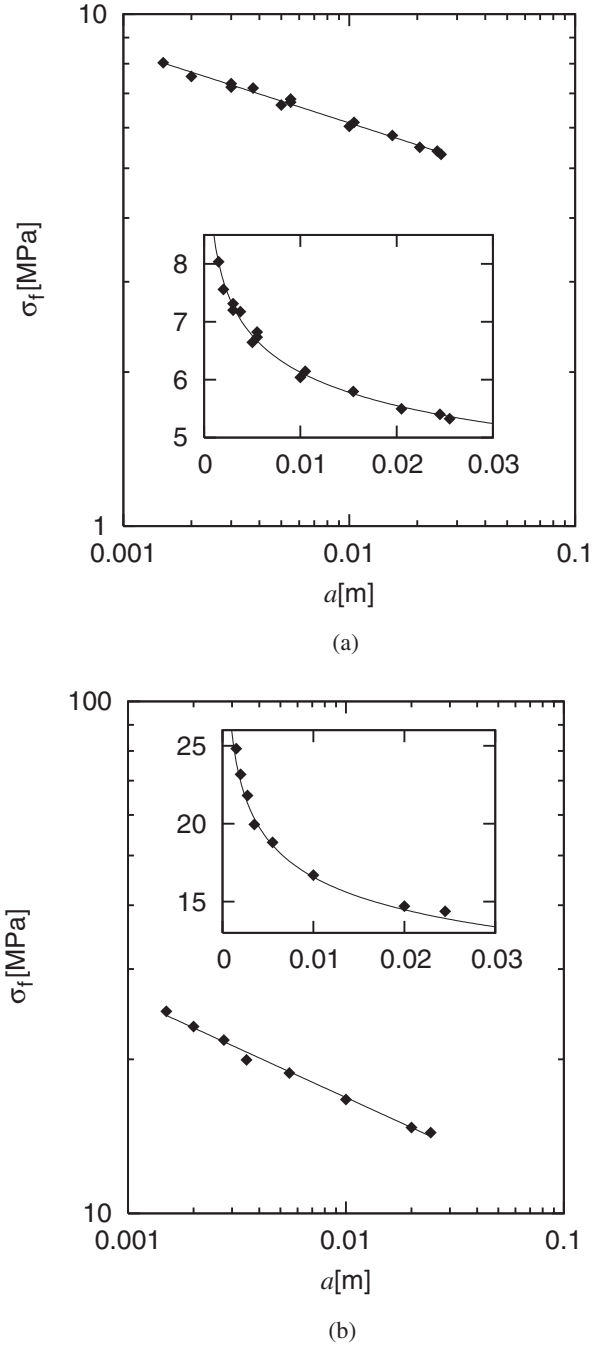

Fig. 2. Failure stress as a function of (half) crack length and the log–log plot obtained from samples, A (a) and B (b). The solid line and curve fit the data using eq. (5).

of crack length. Note that it is difficult to further extend the present experimental range of crack lengths (3 mm to 5 cm) to obtain the desired results. Small line cracks are difficult to make using a knife with high precision. Larger cracks introduce unwanted finite-size effects because the crack size becomes comparable to the dimensions of the sheet: in such a case, the simple theory we developed below becomes invalid.

4. Scaling Arguments

To understand the clear scaling behavior in Fig. 2, we consider a classic relation between the stress σ and the strain ε:

$$\sigma = \begin{cases} E\varepsilon & \varepsilon \ll \varepsilon_0 \\ k\varepsilon^{1/n} & \varepsilon \gg \varepsilon_0 \end{cases}, \tag{1}$$

where E and k are the linear and nonlinear moduli independent of ε, respectively. Here and hereafter, we neglect the tensorial nature of the stress field to focus on scaling laws: σ and ε are the characteristic sizes of the stress and strain tensors, respectively. Under this relation, when the strain ε exceeds the crossover stress ε_0, the stress σ scales nonlinearly as the strain ε with the nonlinear exponent $1/n$.

The energy density defined by

$$w = \int_0^{\varepsilon} \sigma \, \mathrm{d}\varepsilon, \tag{2}$$

exhibits the scaling relation,

$$w = \frac{\sigma^{n+1}}{\kappa}, \tag{3}$$

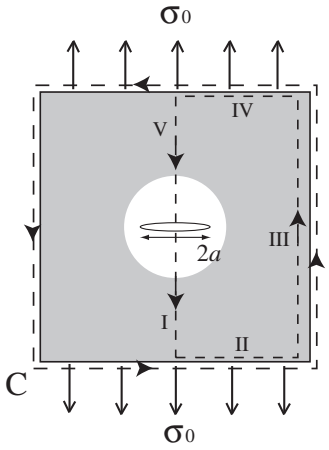

Fig. 3. Plate of nonlinear material with a crack of size a subject to a remote stress σ_0. The plate is assumed to be much larger than the crack size. The stress and strain are relaxed as a whole around the crack compared with the remote homogeneous value of σ_0, as indicated by the gray scale.

if the upper limit of integration, ε, in eq. (2) is sufficiently larger than the crossover stress ε_0. Here, κ is given by

$$\kappa = \left(1 + \frac{1}{n}\right) k^n. \tag{4}$$

Now we extend Griffith's energy balance concept at the critical point of failure[26,27] to nonlinear materials satisfying eq. (3). A (thick or thin) plate of such a material has a line crack (size a) where the size of the plate is much larger than a. This plate is subject to the remote tensile stress σ_0 applied in the direction perpendicular to the line crack (Fig. 3). To open the crack, two surfaces (area $2a$ per unit thickness) should be created. This requires an energy of $4\gamma a$. Here, γ is the fracture surface energy per unit area. Owing to the opening of the crack, the nonlinear elastic energy is reduced (in total) compared with the system without cracks. The amount of energy reduction can be estimated as a characteristic nonlinear elastic energy (which is σ_0^{n+1}/κ per unit volume) localized in a volume (approximately a^2 per unit thickness) because the only length scale available in this problem is the (half) crack length a. Note here that the stress σ_0 is the only available size of the stress (i.e., remote stress). By balancing the reduction energy $\sigma_0^{n+1}a^2/\kappa$ with the energy loss $4\gamma a$, at the failure stress σ_f (i.e., $\sigma_0 = \sigma_f$), we obtain a generalized Griffith formula for the failure stress:

$$\sigma_f = \left(\frac{\kappa\gamma}{\pi a}\right)^{1/(n+1)}. \tag{5}$$

Here, the numerical coefficient is chosen for convenience so that eq. (5) reduces to the standard Griffith's failure formula under the plane stress condition at $n = 1$. Note that the plate in the above argument can be thin or thick. The relation in eq. (5) is valid not only for sheet samples but also for bulk objects, at the level of scaling laws. For bulk materials, the numerical coefficient π in eq. (5) should be replaced with $\pi(1 - \nu^2)$ where ν is Poisson's ratio, for the relation to reduce to the Griffith formula under the plane strain condition at $n = 1$.

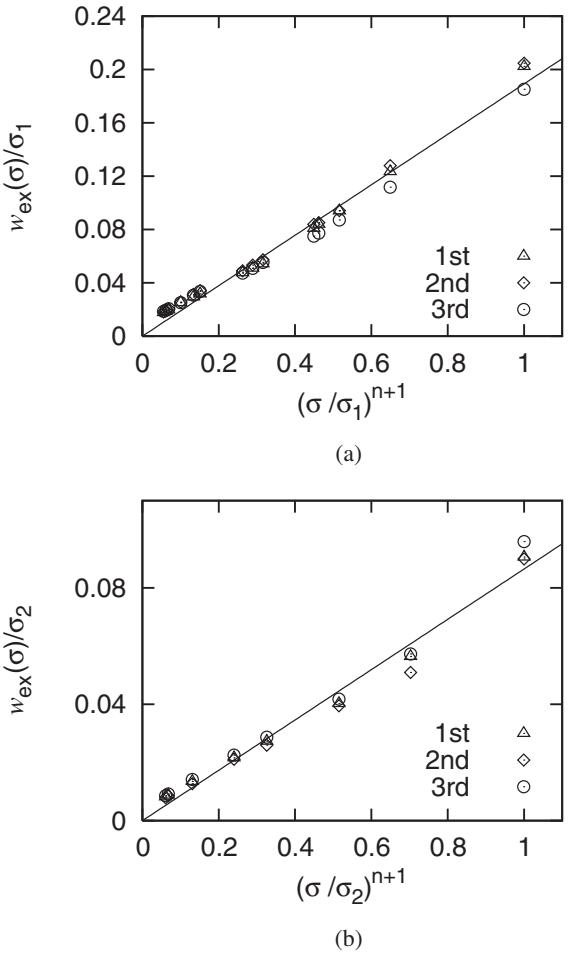

Fig. 4. Energy density $w_{ex}(\sigma)$ vs σ^{n+1} with $n = n_{ex}$ when σ is at the experimentally observed failure stresses for samples, A (a) and B (b). The axes in the top and bottom plots are normalized by $\sigma_1 = 8.04$ MPa and $\sigma_2 = 24.8$ MPa, respectively. The straight line fits the data using eq. (3).

5. Comparison between Experiment and Theory

The experimental relation between σ_f and a shown in Fig. 2 is consistent with eq. (5): the data on the log–log plot are clearly on a straight line over the entire experimental range of σ_f. The slope of the line gives an experimental estimate n_{ex} of the inverse exponent n: we obtain $n_{ex} = 6.04$ and 4.14 for samples A and B, respectively.

The clear power laws demonstrated in Fig. 2 might imply that the relation in eq. (1) with $n = n_{ex}$ is valid and that all the observed strains at the critical point of failure exceed the crossover strain ε_0. This is reasonably well confirmed in the log–log plot in Fig. 1(a) for each sample A: in the log–log plot, the data are represented well by a straight line with a slope $1/n_{ex}$ (this slope is indicated by the solid line) in a high stress range (approximately 5–8 MPa) or in the experimental range of failure strain (approximately 0.05–0.3). For each sample B, in contrast, the straight-line feature in the experimental range of failure strain (0.025–0.15) is less clear [see the log–log plot in Fig. 1(b)]. However, as we see below, eq. (3) is well satisfied for each sample B as well as for each sample A. Note that eq. (5) requires eq. (3) but not necessarily eq. (1).

To confirm eq. (3) for both samples, we show Fig. 4, where w_{ex} and $\sigma^{n_{ex}+1}$ are compared at the experimental

failure stress, $\sigma = \sigma_f$ (σ_f is that appearing in Fig. 2). For a given σ_f, the energy density w_{ex} is obtained by integrating one of the three (first, second, and third) experimental σ–ε curves in Fig. 1 from 0 to ε_f. Here, the upper limit ε_f is the strain on the curve when the stress is σ_f. Accordingly we have three values of w_{ex} for each σ_f. If eq. (3) is satisfied, the ratio $w_{ex}/\sigma_f^{n_{ex}+1}$ for every σ_f is a constant, or the points on the w_{ex}–$\sigma_f^{n_{ex}+1}$ plot are on a straight line. This is well demonstrated in Fig. 4. The constant or the slope of the straight line gives the proportional constant $1/\kappa$ in eq. (3). In this way we can determine the experimental value of κ, i.e., κ_{ex}.

6. Estimation of Fracture Energy

With the use of the κ_{ex} thus obtained, we estimate the fracture energy γ from eq. (5) for samples A and B as 4.2 and 5.5 kN/m, respectively. Here, we assume the numerical factor in eq. (5) to be $\pi^{-1/(n+1)}$ which is exact in the limit $n = 1$. The fracture energy thus obtained is high if we consider only the energy of breaking of chemical bonds (a few N/m). The fracture energy thus obtained reflects extra work for plastic deformation.

In this way, we demonstrate examples in which, even if the stress–strain curve does not exhibit a clear power law over a wide range, a clear scaling relation in eq. (5) can be satisfied over a wide range if eq. (3) is well satisfied. In addition, in such a case, we can estimate the fracture energy.

7. Justification of the Nonlinear Griffith Criterion

We can justify the above derivation of eq. (5) using two other arguments when eq. (3) is satisfied. The first argument is based on the J-integral.[28] The second is based on the HRR crack tip singularity.[22,23]

The J-integral is given as a release rate of elastic energy, i.e., a derivative with respect to the crack length of the elastic potential energy: $J = -\mathrm{d}\Pi/\mathrm{d}a$ with

$$\Pi = \iint_D w \,\mathrm{d}x\,\mathrm{d}y - \int_C \mathbf{T}\cdot\mathbf{u}\,\mathrm{d}s \tag{6}$$

where D is the area of the plate and C is the periphery of D (see Fig. 3). The dot product of the traction force (per area) $\mathbf{T}$ and the deformation $\mathbf{u}$ is integrated along s, i.e., along the path C. Here, the a-dependent part of Π scales as $-w_0a^2$ with $w_0 = \sigma_0^{n+1}/\kappa$ because, when a crack exists, the first double integral is reduced from the value without cracks by the amount w_0a^2 at the level of scaling law. In addition, the second line integral is independent of a under a fixed grip condition. Thus, we obtain $J = -\mathrm{d}\Pi/\mathrm{d}a \sim w_0a$. The same conclusion can be drawn from the standard form,

$$J = \int_\Gamma \left(w\,\mathrm{d}y - T_i \frac{\partial u_i}{\partial x}\,\mathrm{d}s \right) \tag{7}$$

where Γ is a counterclockwise path around a fracture. When we chose the path $\Gamma = \mathrm{I} + \mathrm{II} + \mathrm{III} + \mathrm{IV} + \mathrm{V}$, as in Fig. 3, the sum of the integral $w\,\mathrm{d}y$ along III and (I + V) scales as w_0a, while along II and IV the integral is zero because $\mathrm{d}y$, T_x, and $\partial u_y/\partial x$ are all zero. At the critical point of failure the relation $J \sim w_0a$ thus obtained in two ways reduces to eq. (5) with $\sigma_0 = \sigma_f$ in eq. (3) and $J = 2\gamma$ (γ is the critical value of J). This concludes the justification of eq. (5) based on the J-integral.

The consistency with the HRR stress singularity is confirmed as follows. Dimensionally, the stress field $\sigma(r)$ at a distance from the tip r should be expressed as

$$\sigma(r) = \sigma_0(a/r)^\delta \tag{8}$$

near the tip ($r \ll a$) with an unknown exponent δ. This is because the dimensional quantities available to describe the boundary conditions for this field are only σ_0 and a (because the plate is much larger than a). Note that this form of $\sigma(r)$ recovers to σ_0 at $r \sim a$, as desired. Here, we expect that a simplification occurs in the limit $r \ll a$ at the critical point of failure ($\sigma_0 = \sigma_f$): $\sigma(r)$ at $r \ll a$ is independent of a when $\sigma_0 = \sigma_f$. This is because, on such a small scale, the expression may not depend on a large scale a (compared with r). Namely, we replace σ_0 in eq. (8) with σ_f in eq. (5) and require the expression to be independent of a. We thus determine the exponent δ to be $1/(n+1)$ and recover the HRR relation $\sigma(r) \simeq \sigma_0(a/r)^{1/(n+1)}$ at ($r \ll a$). This concludes the justification based on the HRR theory.

8. Conclusions

We have found a clear scaling relation between failure stress and crack length in nonlinear elastic–plastic materials. This simplification results because finite-size effects are practically suppressed: the thickness and dimensions of samples (approximately 0.01 mm and 50 cm, respectively), and crack sizes (approximately a few cm) are all well separated. Unexpectedly, we find that this scaling relation holds over a wide range even if the stress–strain curve is not well represented by eq. (1). Even in such a case, if eq. (3) is well satisfied, as demonstrated in Fig. 4, we can obtain the fracture surface energy. In other words, the important thing is the scaling property of the energy-stress curve, not of the stress–strain curve, and the former scaling can be expected more easily than the latter: even if the stress–strain relation is not in the form of eq. (1), it often approaches a scaling behavior reasonably well in a high-stress range so that eq. (3) tends to hold well because of the integration in eq. (2). Thus, we expect that this simplification scenario could be widely applicable to many complex and nonlinear materials. In such a case, testing sheet samples (for crack sizes at which failure stress is in the high-stress range) can be useful for determining fracture surface energy even if the stress–strain curve is not in the form of eq. (1). Although this scenario has been demonstrated here for two different samples, it should be desirable to carry out experiments on different classes of materials to confirm the universality of our results, which requires a separate study. We stress here, as stated before, that the nonlinear version of the Griffith formula, eq. (5), confirmed here should be appropriate for thick samples, i.e., for three-dimensional bulk materials.

The nonlinear strain–stress relation discussed in this paper can describe a form of plasticity, known as "deformation plasticity", provided that no unloading occurs. Accordingly, we have developed the above arguments as if we were dealing with a nonlinear elastic model: the discussions above can be regarded for a simple nonlinear material but can also be valid for certain elastic–plastic materials. This is another reason why the theory is in agreement with the results of the experiment on nonlinear plastic materials.

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