

## Inversion Method Based on the Legendre Transformation Applied to Discontinuous Phase Transitions

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We investigated a model of a nematic–isotropic (NI) transition, whose mean-field approximation coincides with Maier–Saupe (MS) theory, by Fukuda’s inversion method (IM), which is based on the Legendre transformation and is similar to a method of symmetry-breaking potential. The first-order approximation of this nonperturbative method neglects fluctuations and reproduces the MS theory, predicting a weakly discontinuous phase transition at the MS transition temperature  $T_{MS}$ . The second-order IM approximation allows us to include fluctuations, which suppress the phase transition.

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### 1. Introduction

The nematic–isotropic (NI) transitions of liquid crystals are usually observed as weak first-order transitions and are often discussed using Maier–Saupe (MS) theory.<sup>1)</sup> The order parameter  $S$  is the average of  $\hat{S}_i$  for an elongated molecule, defined by

$$\hat{S}_i = \frac{3 \cos^2 \theta_i - 1}{2}. \quad (1)$$

Here,  $\theta_i$  is the angle between the direction of the order and that of the long axis of the molecule  $i$ . This theory is a mean-field theory in the sense that all fluctuations are neglected; if we start from an attractive Hamiltonian,

$$h = -\frac{1}{2} \sum_{i,j} u_{ij} \hat{S}_i \hat{S}_j, \quad (2)$$

and then neglect fluctuations,  $\hat{S}_j - S$ , for all the molecules, we arrive at the MS Hamiltonian [note that  $\hat{S}_i \hat{S}_j = \hat{S}_i S + \hat{S}_j (\hat{S}_j - S)$ ]:

$$h_{MS} = -\frac{1}{2} \sum_{i,j} u_{ij} \hat{S}_i S = -N \frac{u}{2} \hat{S} S. \quad (3)$$

This mean-field theory predicts a first-order transition. Here and here after, we consider only the nearest-neighbor (NN) interactions on a lattice of  $N$  sites:  $u_{ij}$  is nonzero only when the pair  $(i, j)$  is an NN pair, and the nonzero value is  $u_0$  ( $\equiv u/z$ ) so that the double summation over  $i$  and  $j$  in the second expression in eq. (3) in fact contains only  $Nz$  terms where  $z$  is the number of NNs per molecule.

In the present work, we begin with the Hamiltonian in eq. (2), and include the effects of fluctuations by Fukuda’s inversion method (IM) developed in field theory,<sup>2)</sup> which is similar to a method of symmetry-breaking potential (see below). This method is useful for improving mean-field theory nonperturbatively; even though the calculation is performed perturbatively (i.e., in a systematic way), each order of the method includes an infinite number of Feynman diagrams, or corresponds to a nonperturbative contribution, while the first-order calculation reproduces the mean-field

theory. It has been applied to a number of problems of *continuous* phase transitions such as symmetry breaking in high-energy physics<sup>3)</sup> as well as in condensed-matter physics including ferromagnetism and superconductivity.<sup>4)</sup> In particular, calculations beyond the mean-field approximation are explicitly considered in the itinerant electron system.<sup>5)</sup> The present study is an application of the method for studying fluctuations neglected in mean-field theory to a *discontinuous* transition. We note here that the mathematical and diagrammatic structure of the method can be used to reformulate simple liquid theory and to renormalize the theory in terms of a three-body density.<sup>6)</sup>

The advantage of applying the Legendre transformation to thermodynamic functions is well-known in statistical physics. Nakano and Hattori indicated that this technique is also powerful in considering the free energy of the order parameter in phase transitions by introducing a symmetry-breaking potential.<sup>7)</sup> In particle physics, essentially the same technique based on the Legendre transformation led to the method of effective action (or effective potential),<sup>8)</sup> frequently used to discuss symmetry-breaking problems in field theory where the statistical average is replaced by the vacuum expectation. The IM is a practically useful technique based on the same technique as the effective action. The IM offers an explicit and general procedure for calculating the free energy (or the effective action) as a function of the order parameter, even though the graph rules are unknown. Historically, the development of IM was principally carried out in the context of field theory, unfortunately without the work by Nakano *et al.* noticed.

Thus, the present study can be regarded as an application of the method of symmetry-breaking potential. In this respect, we note here that Kimura applied this method to a more realistic system of nematic liquid crystal molecules and reproduced the MS theory as a limiting case<sup>9)</sup> (see also ref. 10 for the smectic case). Although the Hamiltonian (thus the expansion parameter) is different, the present work can be regarded as an extension of Kimura’s work, in order to study the physical effects of the higher-order corrections in the same approximation scheme by using a simpler Hamiltonian, where we find a considerable effect: it changes the nature of the transition.

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## 2. Inversion Method for the MS Hamiltonian

In this method, we first modify the Hamiltonian by introducing sources  $J_i$  or symmetry-breaking potentials, which are later set to zero to return to the original theory:

$$h_J = h - \sum_i J_i \hat{S}_i \quad (4)$$

$$= -\frac{1}{2} \sum_{i,j}^{NN} u_{ij} \hat{S}_i \hat{S}_j - \sum_i J_i \hat{S}_i. \quad (5)$$

where  $\sum_{i,j}^{NN}$  is the double summation over NN pairs corresponding to  $Nz$  terms. In our notation, the double counting of each pair by the summation is compensated by the factor  $1/2$  in front of the summation. The free energy as a function of  $J$ ,  $F_J = -\log \text{Tr} e^{-h_J}$ , can be written as

$$F_J = -\log \text{Tr} \exp \left[ \sum_{i,j} \frac{u_{ij}}{2} \hat{S}_i \hat{S}_j + \sum_i J_i \hat{S}_i \right] \quad (6)$$

$$\equiv \sum_{n=0}^{\infty} F_J^{(n)} \quad (7)$$

$$\equiv N f_J, \quad (8)$$

where  $\text{Tr}$  implies the summation over the solid angle for all molecules [see just below eq. (9) for details]. In the above, we used  $kT \equiv 1/\beta$  as the unit of energy; if this unit is not used,  $u_{ij}$  and  $J_i$  should be replaced by  $\beta u_{ij}$  and  $\beta J_i$ , respectively (we use, for example, the reduced temperature  $t$ , defined as  $t \equiv kT/u$  in Fig. 1 below). In addition,  $F_J^{(n)}$  stands for the  $n$ th-order contribution in terms of the coupling coefficient  $u_0$  (the nonzero value of  $u_{ij}$ ), and  $f_J$  is the free energy per site.

This  $n$ th order (in  $u_0$  or  $u$ ) of the free energy  $F_J^{(n)}$  can be calculated perturbatively. The 0th order is readily obtained from eq. (6) by setting  $u_{ij} = 0$  ( $h = 0$ ):

$$\begin{aligned} -F_J^{(0)} &= \sum_i \log \text{tr}_i e^{J_i \hat{S}_i} \\ &\equiv \sum_i M_0(J_i) \end{aligned} \quad (9)$$

where  $\text{tr}_i$  denotes the integral,  $2\pi \int_0^1 dt$  with  $t = \cos \theta_i$ .

The higher orders can be systematically obtained by a diagrammatic technique, as in the high-temperature expansion of local spin systems.<sup>11)</sup> The first-order energy is calculated as

$$-F_J^{(1)} = \frac{u_0}{2} \sum_{i,j}^{NN} M_1(J_i) M_1(J_j), \quad (10)$$

The moment  $M_n(J_i)$  is defined as  $M_n(J_i) \equiv d^n M_0(J_i)/dJ_i^n$ ; the first-order moment is given by

$$M_1(J) = \frac{\text{tr} \hat{S} e^{J \hat{S}}}{\text{tr} e^{J \hat{S}}} \equiv \langle \hat{S} \rangle_J, \quad (11)$$

while the higher-order moments ( $i \geq 2$ ) are given by

$$M_n(J) = \langle (\hat{S} - \langle \hat{S} \rangle_J)^n \rangle_J. \quad (12)$$

In terms of diagrams, the free energy series is expressed as

$$-F_J = \bullet + \bullet \text{---} \bullet + \text{---} \circ \text{---} \circ + \text{---} \bullet \text{---} \bullet + \dots \quad (13)$$

Namely, we can recover the analytical expression of each diagram as follows.

- (1) Assign labels ( $i_1, i_2, \dots$ ) to black circles.
- (2) Associate a factor  $M_n(J_{i_k})$  with a black circle  $i_k$  connected to  $n$  lines.
- (3) Associate a factor  $u_{i_k i_{k'}}$  with a line connecting circles  $i_k$  and  $i_{k'}$ .
- (4) Multiply all the factors thus obtained and take summations over  $i_1, i_2, \dots$  and divide the expression by the symmetry factor.

In fact, we can immediately recover eq. (9) from the first diagram in eq. (13) because the symmetry factor is one for this diagram. For the second diagram, we first obtain

$$\frac{1}{2} \sum_{i,j} u_{ij} M_1(J_i) M_1(J_j), \quad (14)$$

since the symmetry factor here is 2. This expression reduces to right-hand side of eq. (10) because our model considers only NN interactions. As the last example, we give the analytical expression for the fourth diagram:

$$\frac{1}{2} \sum_{i,j,k} u_{ik} u_{jk} M_1(J_i) M_1(J_j) M_2(J_k). \quad (15)$$

The order parameter in the presence of  $J$  is given by  $S_i = -\partial F_J / \partial J_i$ :

$$S_i = S_i^{(0)}[J] + S_i^{(1)}[J] + \dots \quad (16)$$

Here,  $S^{(n)}[J]$  is the  $n$ th order of the coupling constant  $u$  in the perturbative calculation, which is given by

$$S_i^{(n)}[J] = -\frac{\partial F_J^{(n)}}{\partial J_i}. \quad (17)$$

For example, we obtain, from eq. (9),

$$S_i^{(0)}[J] = M_1(J_i), \quad (18)$$

and, from eq. (10),

$$S_i^{(1)}[J] = u_0 M_2(J_i) \sum_j^{NN} M_1(J_j), \quad (19)$$

where  $\sum_j^{NN}$  implies the summation over  $z$  nearest-neighbors (to the  $i$  site). In terms of diagrams, the series is expressed as

$$S_i = \circ + \circ \text{---} \bullet + \text{---} \circ \text{---} \circ + \dots \quad (20)$$

The rules for recovering the analytical expression of each diagram are now as follows:

- (1) Assign labels ( $i_1, i_2, \dots$ ) to black circles, and assign the label  $i$  [the subscript of  $S_i$  in the left-hand side of eq. (20)] to the (only one) open circle.
- (2) Associate a factor  $M_n(J_{i_k})$  with a black circle  $i_k$  connected to  $n$  lines, and assign a factor  $M_{n+1}(J_i)$  with the open circle  $i$  connected to  $n$  lines. Note that a short horizontal bar attached to the left side of the open circle indicates the label  $i$  and is not counted as line.
- (3) Associate a factor  $u_{i_k i_{k'}}$  with a line connecting circles  $i_k$  and  $i_{k'}$ .
- (4) Multiply all the factors thus obtained and take summations over  $i_1, i_2, \dots$  and divide the expression by the symmetry factor. Note that the open circle implies no summation.

As an example, we give the analytical expression for the

third diagram in eq. (20):

$$S_i^{(2)}[J] = \frac{1}{2} \sum_j u_{ij}^2 M_3(J_i) M_2(J_j) \quad (21)$$

$$= \frac{u_0^2}{2} M_3(J_i) \sum_j^{NN} M_2(J_j). \quad (22)$$

Equations (16) or (20), which can be obtained as above, is called *an original series*, and this can be inverted in terms of  $J$  to obtain *an inversion series*:

$$J_i = J_i^{(0)}[S] + J_i^{(1)}[S] + \dots \quad (23)$$

To obtain this series explicitly, we substitute eq. (23) into all  $J$ 's in the right-hand side of eq. (16) and expand around  $J = J^{(0)}$  collecting the same-order terms in  $u$ . Then, we require that the equation should hold at every order of  $u$ . We can thus obtain the following formulae:

$$S_i = S_i^{(0)}[J^{(0)}] = M_1(J_i^{(0)}) \quad (24)$$

$$\sum_j \frac{\partial S_i^{(0)}[J^{(0)}]}{\partial J_j^{(0)}} J_j^{(1)} + S_i^{(1)}[J^{(0)}] = 0 \quad (25)$$

$$\begin{aligned} \sum_j \frac{\partial S_i^{(0)}[J^{(0)}]}{\partial J_j^{(0)}} J_j^{(2)} + \frac{1}{2} \sum_{jk} \frac{\partial^2 S_i^{(0)}[J^{(0)}]}{\partial J_j^{(0)} \partial J_k^{(0)}} J_j^{(1)} J_k^{(1)} \\ + \sum_j \frac{\partial S_i^{(1)}[J^{(0)}]}{\partial J_j^{(0)}} J_j^{(1)} + S_i^{(2)}[J^{(0)}] = 0, \end{aligned} \quad (26)$$

and so on. Here, this inversion is carried out *while regarding  $S_i$  in the right-hand side of eq. (16) as the 0th order of  $u$  (or independent of  $u$ )*. The reason for this will be explained below.

From eq. (24),  $J^{(0)}$  is given as the inverse function of  $S_i^{(0)}[J^{(0)}]$ , while, from the higher-order formula,  $J^{(i)}[J^{(0)}]$  ( $i \geq 1$ ) can be obtained as a function of  $J^{(0)}$ . For example, from eq. (25),  $J_i^{(1)}$  is calculated as

$$J_i^{(1)} = -u_0 \sum_j^{NN} M_1(J_j^{(0)}) = -u_0 \sum_j^{NN} S_j, \quad (27)$$

where the last equality follows from eq. (24). From eq. (26),  $J_i^{(2)}[J^{(0)}]$  is calculated as

$$J_i^{(2)}[J^{(0)}] = -\frac{1}{2} u_0^2 M_3(J_i^{(0)}) \sum_j^{NN} \frac{M_2(J_j^{(0)})}{M_2(J_i^{(0)})}. \quad (28)$$

Here, there is no summation over  $i$ . In this way, we can calculate the inversion series given in eq. (23) up to the desired order, say,  $n = m$ , as a function of  $J^{(0)}$ .

Finally, we return to the original theory by setting  $J = 0$ . This is carried out first by setting all  $J_i$  equal ( $J_i = J$ ) to make  $S_i$  independent of  $i$  ( $S_i = S$ ) and then by setting  $J = 0$ . After these operations, eq. (23) becomes the following equation for  $S$  in the  $m$ th-order approximation of the inversion method ( $m$ th order IM):

$$J^{(0)}(S) + J^{(1)}(S) + J^{(2)}(S) + \dots + J^{(m)}(S) = 0, \quad (29)$$

where  $J^{(0)}(S)$  is formally obtained from eq. (24):

$$J^{(0)}(S) = M_1^{-1}(S) \text{ or } S = M_1(J^{(0)}(S)). \quad (30)$$

In the  $m$ th-order IM, the closed equation (29) for  $S$  is solved using eq. (30). In practice, the inverse function  $M_1^{-1}$  can be

explicitly obtained numerically.

The first-order IM approximation, i.e., eq. (29) with  $m = 1$ , results in solving  $J^{(0)}(S) = -J^{(1)}(S)$ , which reduces to  $M_1^{-1}(S) = uS$  because of eqs. (27) (or  $J^{(1)} = -uS$ ) and (30); in this special case we do not need to know the explicit form of  $J^{(0)}(S)$ ; instead, we obtain

$$S = M_1(uS). \quad (31)$$

This closed or self-consistent equation for  $S$  which determines the order parameter  $S$  exactly reproduces the result of MS theory [see eq. (11)]. Note that the usual first-order approximation in terms of  $u$  [i.e.,  $S = M_1(J) + uM_2(J)M_1(J)$ , from eq. (16)] predicts only the trivial result,  $S = 0$ , because we set  $J = 0$  at the end and we can show that  $M_1(0) = 0$ . In general, we can see that the  $m$ th-order IM approximation is improved compared with the usual (or bare)  $m$ th-order perturbation because of nonperturbative corrections. In terms of Feynman diagrams, the  $m$ th-order IM approximation contains an infinite number of original diagrams, whereas the simple  $m$ th-order calculation includes only a finite number.

In the second-order IM approximation, the closed equation for  $S$ , eq. (29), can be explicitly written as

$$J^{(0)}(S) - uS - \frac{u^2}{2z} M_3(J^{(0)}(S)) = 0. \quad (32)$$

We solved this closed equation for  $S$  numerically for  $z = 6$  [three-dimensional (3D) cubic lattice] as a function of  $u$  or the inverse temperature, by using  $J^{(0)}(S)$  which is also obtained numerically from eq. (24). The result is shown in Fig. 1. We see that fluctuations generally decrease the order in the region below the MS transition temperature  $T_{MS}$ , where  $1/t = u/kT_{MS} \cong 4.5$ ; compared at the same temperature, the nematic order is smaller when the fluctuation is present. However, near and above  $T_{MS}$ , or on the left of  $1/t \cong 4.5$ , no discontinuity can be seen; the discontinuous transition predicted at the mean-field level is modified to a continuous turnover to the disordered phase ( $S = 0$ ). In other words, the phase transition disappears and the nematic order persists at all finite temperatures.

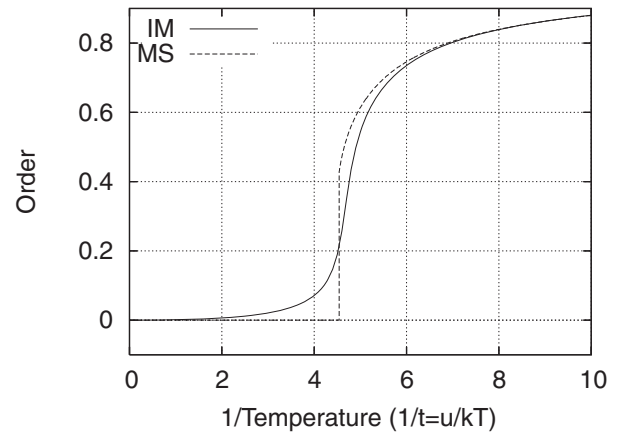


Fig. 1. Nematic order as a function of the inverse of the reduced temperature  $t \equiv kT/u$ . The solid line represents the result of the second-order IM approximation, while the dashed line corresponds to that of the classical MS theory. In the MS theory, the discontinuous transition occurs near the reduced inverse temperature  $u/kT_{MS} = 4.5$ , while the transition disappears in the second-order IM approximation.

To confirm this result from a different viewpoint, we consider the Legendre transformation of the original free energy. The free energy  $F_J$  defined in eq. (6) is a function of the independent variables  $J$  and  $u$ , and then the free energy, defined by

$$F_S = Nf_S = F_J + \sum_i J_i S_i, \quad (33)$$

is independent of  $J$  and is a function of  $S$  and  $u$  with satisfying the relation,  $J_i = \partial F_S / \partial S_i$ . This implies the following: if we solve the relation  $S_i = \partial F_J / \partial J_i$  in terms of  $J$  assuming that  $S$  and  $u$  are independent to obtain  $J = J[S]$  and insert this expression for  $J$  into all  $J$  appearing in  $F_J + J_i S_i$ , then this combination is automatically written in terms of only  $S$  and  $u$ . This is why the inversion is carried out in the above [see the description following eq. (26)] by regarding  $S_i$  in the right-hand side of eq. (16) as the 0th order of  $u$  (or independent of  $u$ ).

Corresponding to the inversion series in eq. (23), the free-energy series is given by

$$F_S = \sum_{n=0} F_S^{(n)}, \quad (34)$$

where the  $n$ th order of the free energy  $F_S^{(n)}$  can be constructed by the integration of the following relation if the explicit form of  $J_i^{(n)}$  is known, which is true for the case of  $n \neq 0$ :

$$J_i^{(n)} = \partial F_S^{(n)} / \partial S_i. \quad (35)$$

The 0th order of the free energy can be calculated, even if the explicit analytical form of  $J_i^{(0)}$  is not available, by the formula

$$F_S^{(0)} = F_J^{(0)} + \sum_i J_i^{(0)} S_i. \quad (36)$$

This formula can be proved by integration by parts of the relation,  $F_S^{(0)} = \int J^{(0)}(dS/dJ^{(0)})dJ^{(0)}$  (ignoring subscript  $i$ ), obtained from eq. (35), together with eq. (17). In fact, differentiating eq. (36) with respect to  $J_i^{(0)}$  leads to  $0 = \partial F_S^{(0)} / \partial J_i^{(0)} + S_i$ , which is equivalent to the inversion formula in eq. (24).

In this way, the free-energy series per site in our model (after setting  $J_i^{(0)} = J^{(0)}$  and thus  $S_i = S$ ) can be calculated explicitly as

$$f_S = f_S^{(0)} + f_S^{(1)} + f_S^{(2)} + \dots, \quad (37)$$

where

$$f_S^{(0)} = -\ln \text{tr} e^{J^{(0)} S} + J^{(0)} S \quad (38)$$

$$f_S^{(1)} = -\frac{u}{2} \langle \hat{S} \rangle_{J^{(0)}}^2 = -\frac{u}{2} S^2 \quad (39)$$

$$f_S^{(2)} = -\frac{u^2}{4z} \langle (\hat{S} - S)^2 \rangle_{J^{(0)}}. \quad (40)$$

In terms of diagrams, the free-energy series is given as

$$-f_S + J^{(0)} S = \bullet + \bullet\text{---}\bullet + \bullet\text{---}\bullet\text{---}\bullet + \dots, \quad (41)$$

where only the irreducible diagrams remain (the fourth diagram in eq. (13) does not contribute). The rules for recovering the analytical expression of each diagram are

the same as those for eq. (13), except that we have to divide the final expression by  $N$  because  $f_S$  is the energy per site and that we have to replace all  $J$  by  $J^{(0)}$  [i.e., the factor  $M_n(J_{i_k})$  for a black circle  $i_k$  connected to  $n$  lines is replaced by the factor  $M_n(J_{i_k}^{(0)})$ ]. Thus, all the diagrams are functions not of  $J$  but of  $J^{(0)}(S)$ , making each diagram a function of  $S$ . In fact, from the diagram rules we obtain, from the first diagram,

$$\frac{1}{N} \sum_i M_0(J_i^{(0)}), \quad (42)$$

which reduces to the first term in eq. (38) because of the definition of  $M_0$  given in eq. (9). For the second diagram, we first obtain eq. (14) divided by  $N$  but with  $M_1(J_i)M_1(J_j)$  replaced by  $M_1(J_i^{(0)})M_1(J_j^{(0)})$ , which reduces to eq. (39) using eq. (30). By the same rules, the third diagram can be recovered as

$$\frac{1}{4N} \sum_{i,j} u_{ij}^2 M_2(J_i^{(0)})M_2(J_j^{(0)})$$

which reduces to eq. (40) by eqs. (12), (11), and (30).

The  $m$ th-order IM approximation for the free energy is defined by  $f_S = \sum_{n=0}^m f_S^{(n)}$  with eqs. (24) or (30). In particular, the first-order IM free energy is given from eq. (37) as

$$f_{MS} = \log \frac{\text{tr} 1}{\text{tr} e^{uSS}} + uS^2/2 \quad (43)$$

by putting  $f_{MS} = 0$  at  $S = 0$ . The energy in eq. (43) is the well-known free energy obtained in MS theory. Actually, if we set  $J = 0$  in the relation  $J = \partial f_{MS} / \partial S$  with  $f_{MS}$  given in eq. (43), we can recover eq. (31) as desired.

In the second-order IM approximation, the free energy is given by  $f_{IM} = f_S^{(0)} + f_S^{(1)} + f_S^{(2)}$  with eqs. (38)–(40). Again, if we put  $J = 0$  in  $J = \partial f_{IM} / \partial S$  for this  $f_{IM}$ , we obtain eq. (32) as expected.

The free energy as a function of  $S$  is plotted in Figs. 2 and 3, where the numerical values of  $J^{(0)}$  are again employed. These plots are consistent with the results in Fig. 1; we see that the global minimums of the free energy in Figs. 2 and 3 agree with the values of order parameter  $S$  at the corresponding temperatures in Fig. 1.

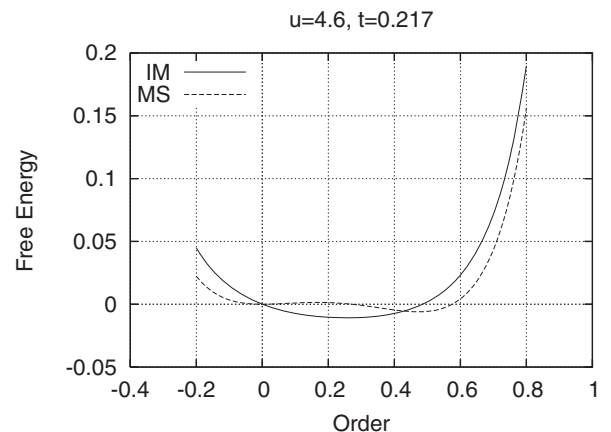


Fig. 2. Free energy as a function of nematic order at  $u = 4.6$ . The solid line represents the result of the second-order IM approximation, while the dashed line shows that of the classical MS theory.

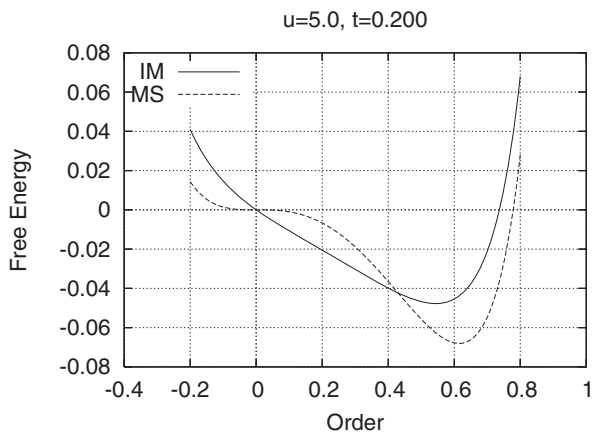


Fig. 3. Free energy as a function of nematic order at  $u = 5.0$ .

### 3. Conclusion and Discussion

We studied the effect of fluctuations on the mean-field result by the inversion method (or the method of symmetry-breaking potential) in a model of a discontinuous transition. We found that the model, whose mean-field approximation predicts a first-order transition, is changed into a continuous turnover to the disordered phase when fluctuations are included. This might be related to the weakly discontinuous nature of nematic–isotropic transitions in liquid crystals. However, we have to be rather careful in interpreting the present result in the context of understanding the N–I transition of liquid crystals: (1) There is a possibility that the next-order or even higher-order calculations might predict discontinuous transitions again which is suppressed in the second-order IM approximation.<sup>12)</sup> (2) The Hamiltonian in eq. (2) breaks the rotational symmetry, which has been pointed out to be important in describing the N–I transition in real systems. In the MS model, Ypma and Vertogen applied a generalized Bethe approximation to show the disappearance of the N–I phase transition<sup>13)</sup> and Priest gave arguments to support its disappearance in the  $n$ -site cluster expansion<sup>14)</sup> proposing to study rotationally invariant analogues of the original (mean-field) MS model.

In terms of understanding the mathematical nature of the inversion method, to proceed to the next order of our calculation would be worthwhile, which would require a separate publication. Furthermore, direct renormalization group (RG)<sup>15)</sup> studies would be desirable on the lattice MS model, particularly in the context of understanding the mathematical aspects of IM for discontinuous transitions. Such a study is not currently available, although RG studies exist on closely related models such as the Landau–de Gennes Q models.<sup>16–18)</sup> For such a purpose, we could start from an exact functional integral representation of the lattice MS Hamiltonian via a Hubbard–Stratonovich transformation<sup>19)</sup> or by defining an appropriate projection operator to perform a real-space RG transformation directly on the lattice MS Hamiltonian.

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