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Simple Network Model for Reinforcement of Materials with Voids

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In nature, some materials with voids are strong and tough. For example, voids in apples contribute to their toughness,[1] the exoskeleton of sea cucumbers has many voids that contribute to the highly resilient texture that Japanese people enjoy eating,[2] and biosilica, known for its remarkable porous structure, is reinforced by the structure.[3] Inspired by such biological examples, we here study the strength of materials with voids, or porous materials. To discuss the strength of materials, we exploit a well-established strategy in fracture mechanics:[4,5] we consider a sample with a macroscopic crack (larger than the minimum length scale below which continuum mechanics breaks down) and examine how the sample maintains its strength.

Materials start to break when local stress exceeds a certain critical value, which we call the local critical stress. When a material with a crack in the center is under tension, the stress is concentrated at the crack tips. Accordingly, the material starts to fail at the crack tips when the tip stress exceeds the local critical stress. The stress applied at the sample edges, away from the crack, at this critical state, is called the failure stress, which is smaller than the local critical stress. Note that materials are regarded as strong when the remotely applied failure stress is large: the failure stress is a measure of the strength of materials.

In this note, we examine how the failure stress changes with the size of voids, while the bulk elasticity and local critical stress are fixed. Experimentally, this corresponds to determine the optimum size of voids in terms of the material strength by testing samples with voids that are made of the same material (and thus the local critical stresses are the same) and that possess the same volume fraction (and thus the bulk elasticities are the same). We surprisingly find a clear scaling law that implies that, in term of the material strength (i.e., in order to increase the failure stress), it is better to increase the void size.

As an approach to solve the problem, we consider a simple two-dimensional network motivated by simple examples explaining the toughness and strength of biomaterials.[6–8] In the network shown in Fig. 1(a), $N \times N$ beads are connected to their nearest neighbors with nonlinear springs whose spring constant and natural length are k and l, respectively (the undeformed system size is $Nl \times Nl$). Nonlinear springs are introduced to satisfy the stress-strain relation $\sigma = E\varepsilon^{1/n}$, where σ and ε are the stress and strain, respectively, with $E = kl^{1/n-1}$ (stress in this paper is two-dimensional, i.e., force divided by mesh size). In this model, the length of the springs, l, is regarded as the size of the voids in the structure. To change the size of the voids, we consider networks with different mesh sizes by setting the length of the springs to $d = ml$ (by changing m but with l fixed) [see Fig. 1(b) for $m = 2$]. To maintain the same bulk elasticity for different m values, we only have to set the spring constant to $k_m = km^{1-1/n}$; this corresponds to considering that a spring of length ml is composed of an m-serial connection of an m-parallel connection of springs of length l; in this arrangement, when the system size is fixed to $Nl \times Nl$ (the number of beads is $N/m \times N/m$), the total number of springs of length l in the system is always the same [$\sim 2(N/m)^2m^2 \sim 2N^2$] when the system is large ($N \gg 1$). Experimentally, this corresponds to maintain the same volume fraction of material, i.e., to give the same bulk elasticity, for materials with different void sizes.

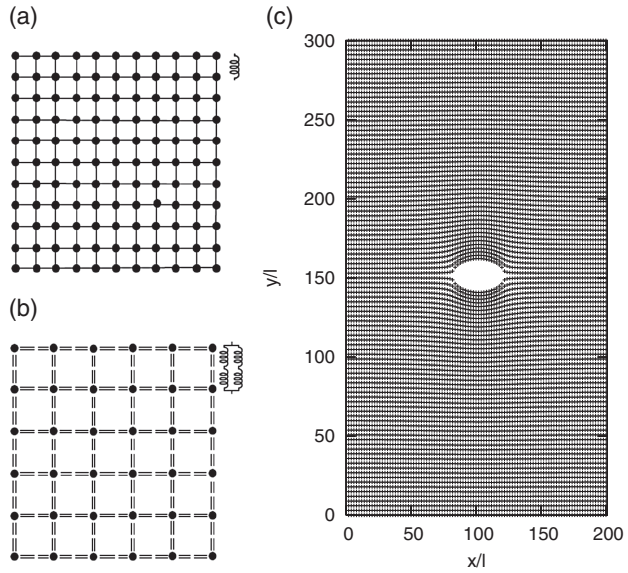

Fig. 1. Network model of materials with voids. (a) Standard mesh size, $d = l$. (b) Varied mesh size, $d = ml$ with $m = 2$. (c) Network with a crack stretched in the direction perpendicular to the crack in the equilibrium state ($m = 2$, $a = 40l$, and $\varepsilon = 0.5$).

As shown in Fig. 1(c), we suppress springs near the center to mimic a line crack of length a and stretch the system in the direction perpendicular to the crack. The equilibrium positions are calculated by minimizing the total energy of the springs by the conjugate gradient method.[9] In the minimization, we set the spring constants of the springs located at the line crack to zero, fix the beads at the top and bottom edges to apply fixed strains at the edges, and use the equilibrium positions in the absence of the crack as the initial condition.

To numerically determine the failure stress, we gradually increase the strain at the sample edges until the stress at the crack tips (where the maximum stress appears in the equilibrium state) reaches the local critical stress σ_c, which is independent of m: the failure stress is defined as the stress at the edges at this critical state. Technically, in terms of the force on the springs, this amounts to the following: we introduce a critical force mf_c for a spring of length ml (an m-serial connection of an m-parallel connection of springs of length l), and regard σ_c as the critical force divided by the mesh size, $mf_c/ml = f_c/l$, which is independent of m. For numerical convenience, σ_c is determined by setting f_c to the value when the strain of a spring of length l reaches 0.5.

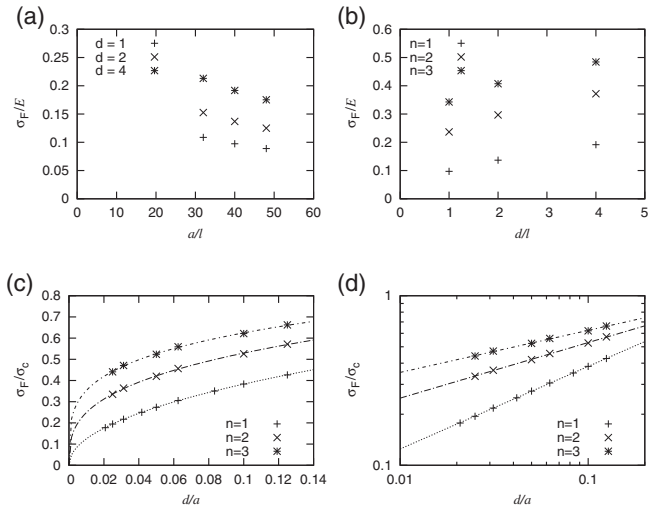

Fig. 2. (a) Failure stress σ_F vs crack length a in the linear system ($n = 1$). (b) σ_F vs void size d for $a = 40l$ in nonlinear systems ($n = 1$, 2, and 3). (c) Failure stress σ_F normalized by critical local stress σ_c as a function of void size d renormalized by crack size a. (d) Same plot in log–log scale.

In Fig. 2(a), the numerically obtained failure stress σ_F is given as a function of the crack size a ($a = 32l$, $40l$, and $48l$) in networks with an unstretched size of $200l \times 200l$ for different mesh sizes of $d = l$, $2l$, and $4l$ ($m = 1$, 2, and 4) for linear-springs ($n = 1$). The failure stress monotonically decreases as the crack size is increased for a fixed mesh size and increases with the mesh size for a given crack size. In Fig. 2(b), σ_F is similarly shown to increase with d for the crack size $a = 40l$ in nonlinear systems ($n = 1, 2, 3$).

In the present calculations, the important lengths are a and d because the system size is much larger than these two lengths (l only defines the unit of length). Thus, if a simple scaling law exists for the failure stress σ_F, it is likely that σ_F can be completely described by these length scales and the only characteristic scale for the stress in the present case, i.e., σ_c (E only defines the unit of stress). In this way, we expect a scaling law in the form $\sigma_F \sim \sigma_c(d/a)^\alpha$.

Motivated by the above naive expectation, we collect all the data in Figs. 2(a) and 2(b) together with other additional data to make a plot with the renormalized axes σ_F/σ_c and d/a to find a master curve for each n, as shown in Fig. 2(c); Data for different mesh sizes ($m = 1$, 2, and 4) and different crack sizes ($a = 32l$, $40l$, and $48l$) for a given n (either 1, 2, or 3) convincingly collapse onto a single curve. Furthermore, from the corresponding log–log plot given in Fig. 2(d), we obtain the slopes of 0.4950, 0.3274, and 0.2494 for $n = 1$, 2, and 3, respectively, which imply that the slopes are given by $1/(n + 1)$. We thereby give the following scaling law:

$$\sigma_F/\sigma_c \sim (d/a)^{1/(n+1)}. \quad (1)$$

This scaling law states that the failure stress increases with the mesh size, indicating that materials with voids for a given volume fraction are reinforced when the void size is increased. Practically, the size of voids in materials is limited by other factors. For example, for continuum materials, the void size may have to be much smaller than their dimensions, and when materials should be transparent, the void size should be smaller than the wavelength of light in question.

The predicted law is consistent with the well-known crack tip singularity for nonlinear elastic systems, expressed as $\sigma(r)/\sigma_0 = (a/r)^{1/(1+n)}$, which was derived in Refs. 10 and 11 where r is the distance from the crack tip. This implies that, mathematically, the stress diverges at the crack tip. However, in reality, at a certain scale, d_0, the continuum description breaks down. The simplest possibility is that the maximum stress σ_M appearing at the crack tips is given by the above expression in which r is cut off at d_0, i.e., $\sigma_M \equiv \sigma(r = d_0) \sim \sigma_0(a/d_0)^{1/(1+n)}$. In fact, this possibility has been confirmed in some cases.[12,13] This formula means that the maximum stress appearing at the crack tips decreases with increasing void size, thereby suggesting the possible reinforcement of materials with voids. This suggestion is confirmed in the form of another more direct scaling law in the present study.

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