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EFFECTIVE ACTIONS OF LOCAL COMPOSITE OPERATORS: THE CASE OF φ^4 THEORY, THE ITINERANT ELECTRON MODEL, AND QED

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The effective action $\Gamma[\phi]$, defined from the generating functional $W[J]$ through the Legendre transformation, plays the role of an action functional in the zero temperature field theory and of a generalized thermodynamical function(al) in equilibrium statistical physics. A compact graph rule for $\Gamma[\phi]$ of a local composite operator is given in this paper. This long-standing problem of obtaining $\Gamma[\phi]$ in this case is solved directly without introducing the auxiliary field. The rule is first deduced with help of the inversion method, which is a technique for making the Legendre transformation perturbatively. It is then proved by using a topological relation and also by the summing-up rule. The latter is a technique for making the Legendre transformation in a graphical language. In the course of proof a special role is played by $J^{(0)}[\phi]$, which is a function(al) of the variable ϕ and is defined through the lowest inversion formula. Here $J^{(0)}[\phi]$ has the meaning of the source J for the noninteracting system. Explicitly derived are the rules for the effective action of $\langle\varphi(x)^2\rangle$ in the φ^4 theory, of the number density $\langle n_{\mathbf{r}\sigma}\rangle$ in the itinerant electron model, and of the gauge-invariant operator $\langle\bar{\psi}\gamma^\mu\psi\rangle$ in QED.

1. Introduction

The effective action $\Gamma[\phi]$ or thermodynamical function introduced by Legendre transformation is a very convenient tool in various fields of physics. Actually this fact has long been realized in condensed matter physics as well as in particle physics.[1]

In spite of its widespread use, the precise rule for constructing the effective action for a *local composite field* seems to be difficult although the graphical rules for an elementary field and for nonlocal composite fields up to four-body operators are already known.[2–6] The study of the effective action for a local composite operator amounts to rewriting the theory in terms of physical variables such as the expectation values of the number density operator, spin density operator, local gauge-invariant operator, etc. Thus the importance of the investigation cannot be

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overestimated. In the following we deal with three examples — the effective action of the $\varphi(x)^2$ operator in the φ^4 theory, a generalized free energy as a function(al) of the spin and number density in the itinerant electron model, and the effective action of the $\bar{\psi}(x)\gamma^\mu\psi(x)$ operator in QED (where ψ is the electron field).

In some cases the hard problem of obtaining the effective action of local composite operators has been avoided by Hubbard–Stratonovich transformation[7] or by introducing an auxiliary field.[8] In such a formulation, the auxiliary field is not equal to the local composite operator if one deals with the off-shell quantities and extra work is needed to extract the physical on-shell quantities, which are directly related to the original local composite operator. In addition, there are important operators, such as $\bar{\psi}(x)\psi(x)$ in QED or QCD in the study of the chiral symmetry breaking, which cannot be dealt with by the auxiliary field technique. This article can be regarded as the first step toward these cases.[a]

In the following we deal with the local composite operator itself without introducing an auxiliary field and explicitly derive the graphical rule for the effective action. Difficulty is solved by using the inversion method.[10–14] As seen below, in our formalism the quantity $J^{(0)}(\phi)$ plays an essential role since we rely on the inversion method so that it is natural that the present work is different from Ref. 15, which explicitly deals with the Gross–Neveu model in a way suitable for $1/N$ expansion.

In this article we are concerned with only the combinatorial aspect of the subject. For the field-theoretical cases there remain the crucial issues of renormalization of composite operators. This important aspect of the problem is mentioned in the Discussion (Sec. 5). A complete study of the problem is very important and needs a separate investigation. On the other hand, in the case of the system defined on a lattice, the results are readily applicable to the actual physics[14] without being influenced by the divergence problem.

We emphasize here that the analysis of the Feynman diagrams appearing in Γ is no doubt indispensable for getting the solution of the renormalization problem about the effective action of the local composite operator.

For later discussion let us define the effective action $\Gamma[\phi]$ explicitly. For the zero temperature case it is introduced through a generating functional $W[J]$ with a source J coupled to some operator $\hat{O}$; $e^{iW[J]} = \langle 0|e^{iJ\hat{O}}|0\rangle$. Here $|0\rangle$ represents the ground state. Then a dynamical variable ϕ is defined as $\phi = \frac{\delta W}{\delta J} \equiv \langle \hat{O}\rangle^J$ and the effective action, which is a functional of ϕ, is given by $\Gamma[\phi] = W[J] - J\phi$ with $-J = \frac{\delta\Gamma}{\delta\phi}$. Here J *is given by a functional of* ϕ by inverting $\phi = \frac{\delta W}{\delta J}$. For simplicity we have considered the x-independent variables J and ϕ since it is straightforward to extend to the local variables $J(x)$ and $\phi(x)$. We have called a function of J or ϕ a *functional*, as we will do in what follows, so that we can recover the x dependence freely. In equilibrium statistical physics W corresponds to the thermodynamical

[a]Actually, the results presented here have been developed to be applied to the gauge-invariant study of the strong coupling phase of massless QED and positronium states.[9]

potential Ω. For instance, $\hat{O}$ is chosen to be the total number operator N; then Γ corresponds to the Helmholtz free energy F and J is the chemical potential $\mu(N)$.

The essential step of Legendre transformation is to invert the relation $\phi = \frac{\delta W}{\delta J}$ in terms of J. The inversion method enables us to write down the explicit form of J in terms of ϕ by perturbative calculation. The lowest relation of the method defines the functional $J^{(0)}[\phi]$, which is the source as a functional of ϕ in a noninteracting system. As will become clear, it is $J^{(0)}[\phi]$ that plays a fundamental role in deriving $\Gamma[\phi]$. In fact it turns out that, by the inversion method, $\Gamma[\phi]$ in the case of a local composite field is obtained as a class of *irreducible* graph in a certain sense (plus simple terms) as a functional of $J^{(0)}[\phi]$ rather than ϕ [for the φ^4 theory, see (2.69) with (2.46) or (2.90)]. In other words, all the functional dependence on ϕ is through $J^{(0)}[\phi]$. This point is in remarkable contrast to the rules for the effective action of an elementary field and nonlocal composite operators where the rule is based on ϕ itself. This may be the reason why the problem of the local composite operator is difficult and has been unresolved. The use of $J^{(0)}[\phi]$ naturally comes out in the formulation through the inversion method.

In order to explain the inversion method[10–14] (again for the simple case of the x-independent variables J and ϕ), we assume that the theory has a coupling constant λ. Then the expectation value $\phi = \langle\hat{O}\rangle^J$ is calculated in the presence of J through the Feynman rule [like (2.9)] to get a series expansion

$$\phi = \sum_{n=0}^{\infty} \phi^{(n)}[J]\,, \tag{1.1}$$

where $\phi^{(n)}[J]$ is the nth order of λ by regarding J as independent of λ. This relation can be inverted to give

$$J = \sum_{n=0}^{\infty} J^{(n)}[\phi]\,, \tag{1.2}$$

where $J^{(n)}[\phi]$ is the nth order of λ. To obtain the explicit form of $J^{(i)}[\phi]$ as a functional of ϕ we first assume (1.2) and get

$$\phi = \sum_{n=0}^{\infty} \phi^{(n)} \left[\sum_{n=0}^{\infty} J^{(n)}[\phi]\right] \tag{1.3}$$

$$= \phi^{(0)}\left[J^{(0)}[\phi] + J^{(1)}[\phi] + \cdots\right] + \phi^{(1)}\left[J^{(0)}[\phi] + J^{(1)}[\phi] + \cdots\right] + \cdots \tag{1.4}$$

or

$$\phi = \phi^{(0)}\left[J^{(0)}[\phi]\right] + \phi^{(0)\prime}\left[J^{(0)}[\phi]\right]J^{(1)}[\phi] + \cdots + \phi^{(1)}\left[J^{(0)}[\phi]\right] + \cdots\,, \tag{1.5}$$

where $\phi^{(0)\prime}[J] = \frac{\delta\phi^{(0)}[J]}{\delta J}$. The inversion is made *by regarding ϕ as independent of λ*, namely as of order $\lambda^0 = 1$. Then an explicit form for $J^{(n)}[\phi]$ is known successively up to the desired n by writing down the nth order of (1.5); $\phi = \phi^{(0)}[J^{(0)}[\phi]]$,

$J^{(1)} = -\phi^{(1)}[J^{(0)}[\phi]]/\phi^{(0)\prime}[J^{(0)}[\phi]], \ldots$. Regarding ϕ as independent of λ just corresponds to making the Legendre transformation from J to ϕ (see App. A). The extension of the above formula to the case of local variables $J(x)$ and $\phi(x)$ can be done merely by recovering the x dependence and appropriate space–time integrals. We will see that the series expansion (1.5) in the graphical form is directly given by (2.10). An explicit form of $J^{(0)}[\phi]$ may not be obtainable in the cases studied in this paper because $J^{(0)}[\phi]$ is defined by the inverse of a known functional $\phi^{(0)}[J]$ or $J^{(0)}[\phi] = \phi^{(0)-1}[\phi]$. However, examples in which $J^{(0)}[\phi]$ is explicitly obtained are dealt with in App. C. But this is not necessarily an obstacle; rather, it may be a merit in actual calculation in some cases. An explicit instance in this respect has been provided for the case of the itinerant electron model.[14] In other cases it is more convenient to change the dynamical variable; $\phi \to J^{(0)}[\phi]$, as in Ref. 12.

In Sec. 2 the case of the φ^4 theory is discussed in detail as the simplest example and also as a prototype for the subsequent two models. First we try to deduce the rule and arrive at the propositions to be proved later. Explicit rules are given in the form of Proposition A2 with A1′ or Proposition A3′ below. In the second subsection we rigorously prove these propositions in two ways: by the use of a topological relation and by the summing-up rule.[16] In Sec. 3 the case of the itinerant model is studied as an example of the free energy of condensed matter physics. A more model-specific study of the case has been carried out[14] to give a systematic improvement of the Stoner theory and to obtain the results similar to the SCR theory by Moriya and Kawabata.[17] The last example of QED is given in Sec. 4, which can be the first step toward a gauge-invariant study of the gauge field theory. A discussion on the renormalization problem is given in Sec. 5, as stated before. Appendix A gives the reason why ϕ is to be considered as independent of λ in the process of inversion in a way different from the one given in the literature. In App. B the Feynman rules which are necessary for our discussion are given in detail because the symmetry factors play an important role in the deduction of the rule. Appendix C reproduces the known rules of the effective actions for an elementary field and nonlocal two-body composite operators by the inversion method. In these cases $J^{(0)}[\phi]$ can be explicitly given, as stated before. In App. D we review the path integral technique for the fermion coherent state used in Sec. 3.

2. The Case of φ^4 Theory

As the simplest example we consider the effective action for the expectation value of a *local composite operator* $\varphi(x)^2$ in the φ^4 theory — we take $\Gamma[\phi]$ with the local variable $\phi(x) \propto \langle \varphi(x)^2 \rangle$.

Let us introduce the generating functional $W[J]$ in the path integral representation as follows:

$$e^{iW[J]} = \int \mathcal{D}\varphi e^{iS[\varphi, J]}\,, \tag{2.1}$$

$$S[\varphi, J] = -\frac{1}{2}\int d^4x d^4y \varphi(x) G^{-1}(x,y)\varphi(y)$$
$$-\frac{\lambda}{4!}\int d^4x \varphi(x)^4 + \frac{1}{2}\int d^4x J(x)\varphi(x)^2\,, \tag{2.2}$$

$$G^{-1}(x,y) = (\Box + m^2)\delta^4(x-y)\,, \tag{2.3}$$

where $\int \mathcal{D}\varphi$ denotes the functional path integration by the field φ. Note here that an x-dependent local external source $J(x)$ is coupled to the local composite field operator $\varphi(x)^2$. In general, apart from the situation where the symmetry of the Lagrangian preserves it, the vacuum expectation value of the composite operator $\varphi(x)^2$ is nonvanishing even if the system is not in the condensed phase. This is the case with the φ^4 theory. The true condensation should then be the difference between $\langle\varphi(x)^2\rangle_{J=0}$ evaluated at the perturbative vacuum and the nonperturbative (i.e. condensed) vacuum. Thus if one wants to write the effective action as a functional of the condensation in the above sense, it may be appropriate to change the source term from $J(x)\varphi(x)^2$ to $J(x)\big(\varphi(x)^2 - \langle\varphi(x)^2\rangle^p_{J=0}\big)$, where $\langle\varphi(x)^2\rangle^p_{J=0}$ is evaluated at the perturbative vacuum. But we do not introduce the source term like this for the following two reasons:

(1) We could know the value of the condensation in the above sense by comparing the two solutions for the stationary equation of the effective action. One of them is of course $\langle\varphi(x)^2\rangle^p_{J=0}$.
(2) We are mainly interested in the mathematical structure of our problem, so that it is wise to choose the simpler option.

Hereafter we frequently use the notation in which the space–time indices and their integrations are omitted if it causes no ambiguity. For example, $S[\varphi, J]$ in (2.2) is denoted as

$$-\frac{1}{2}\varphi G^{-1}\varphi - \frac{\lambda}{4!}\varphi^4 + \frac{1}{2}J\phi$$

in this symbolic notation.

It is straightforward to get the graphical rule for $W[J]$. We note here that different rules are obtained depending on how much of J is absorbed in the propagator. In this paper both of the two diagrammatic rules (2.4) and (2.5) are used:

$$W[J] - W_0 = -\frac{1}{2i}\operatorname{Tr}\ln G_J^{-1} + \frac{1}{i}\langle e^{-\frac{i\lambda}{4!}\varphi^4}\rangle_{G_J}\,, \tag{2.4}$$

i.e. the sum of all the connected vacuum graphs built with the four-point vertex $-\lambda$ and the propagator G_J, and

$$W[J] - W_0 = -\frac{1}{2i}\operatorname{Tr}\ln[G^{(0)}]^{-1} + \frac{1}{i}\langle e^{-\frac{i\lambda}{4!}\varphi^4 + \frac{i}{2}(J^{(1)}+J^{(2)}+\cdots)\varphi^2}\rangle_{G^{(0)}}\,, \tag{2.5}$$

i.e. the sum of all the connected vacuum graphs constructed out of the four-point vertex $-\lambda$, the two-point vertex $J^{(i)}$ with $i \geq 1$, and the propagator $G^{(0)}$. Here the propagators are defined as (with obvious symbolic notation)

$$G_J^{-1} = \Box + m^2 - J\,, \qquad [G^{(0)}]^{-1} = \Box + m^2 - J^{(0)}\,. \tag{2.6}$$

W_0 is the trivial J-independent part of W and Tr represents the functional trace. The first term on the right hand side of (2.4) or (2.5) (Tr ln term) is usually denoted by a circle in graphical representation and, in this paper, is called a *trivial skeleton* (the definition of the skeleton itself is given below). Furthermore the notation of the form $\langle O[\varphi]\rangle_A$ means the summation of all the possible connected Wick contractions of the operators contained in $O[\varphi]$ by using A as propagators, i.e.

$$\langle O[\varphi]\rangle_A = \left.\frac{\int \mathcal{D}\varphi e^{iS_0} O[\varphi]}{\int \mathcal{D}\varphi e^{iS_0}}\right|_{\text{conn.}}\,, \quad \text{with} \quad S_0 = -\frac{1}{2}\varphi A^{-1}\varphi\,. \tag{2.7}$$

Throughout this paper we frequently employ this notation from which the *weights of graphs* are explicitly obtained. Remember that the original notation $\langle\varphi(x)^2\rangle$, $\langle\varphi(x)^2\rangle^J$ implies, however, the full order expectation value. The rule (2.5) contains two-point vertices of $J^{(i)}\varphi^2$ ($i \geq 1$) because the absorption of J into the propagator is not complete.

Now the expectation value of the local composite field will be called $\phi(x)$; specifically,

$$\phi(x) = \frac{\delta W}{\delta J(x)} \equiv \frac{1}{2}\langle\varphi(x)^2\rangle^J\,. \tag{2.8}$$

With the notation (2.7) the graphical rules corresponding to (2.4) and (2.5) are summarized as

$$\phi = \left\langle \frac{1}{2}\varphi^2 e^{-\frac{i\lambda}{4!}\varphi^4} \right\rangle_{G_J}\,, \tag{2.9}$$

i.e. the sum of all the connected graphs with one external point (where two propagators meet) built with the four-point vertex $-\lambda$ and the propagator G_J, and

$$\phi = \left\langle \frac{1}{2}\varphi^2 e^{-\frac{i\lambda}{4!}\varphi^4 + \frac{i}{2}(J^{(1)}+J^{(2)}+\cdots)\varphi^2} \right\rangle_{G^{(0)}}\,, \tag{2.10}$$

i.e. the sum of all the connected graphs with one external point (where two propagators meet) built with the four-point vertex $-\lambda$, the two-point vertex $J^{(i)}$ ($i \geq 1$), and the propagator $G^{(0)}$.

To rewrite the theory in terms of this dynamical variable ϕ instead of J, we introduce as usual the effective action of ϕ through Legendre transformation:

$$\Gamma[\phi] = W[J] - \int d^4x J(x)\phi(x) \equiv W[J] - J\phi\,, \tag{2.11}$$

with an identity

$$-J(x)=\frac{\delta\Gamma[\phi]}{\delta\phi(x)}\,. \tag{2.12}$$

It is convenient to introduce $\Gamma^{(n)}$, which is the nth order in λ, or

$$\Gamma=\sum_{n=0}^{\infty}\Gamma^{(n)}\,. \tag{2.13}$$

Then we see in Subsec. 2.1.2 that $\Gamma^{(0)}$ and $\Gamma^{(1)}$ are explicitly given by (suppressing the space–time integration)

$$\Gamma^{(0)}=-J^{(0)}[\phi]\phi-\frac{1}{2i}\mathrm{Tr}\ln[G^{(0)}]^{-1}\,, \tag{2.14}$$

$$\Gamma^{(1)}=-\frac{1}{2}\lambda\phi^2\,. \tag{2.15}$$

In this case of the φ^4 theory, $J^{(0)}[\phi]$ is defined through

$$\phi(x)=\frac{1}{2i}G^{(0)}(x,x)=\frac{1}{2i}\left(\frac{1}{\Box+m^2-J^{(0)}[\phi]}\right)_{xx}\,, \tag{2.16}$$

which is to be proved in Subsec. 2.1.1. We emphasize here that although the right hand side is denoted by a single graph of (2.31), ϕ on the left hand side is a full order quantity, suggesting that $J^{(0)}[\phi]$ has full order information. The central part of our study is that for the remaining part of Γ, which is called $\Delta\Gamma$,

$$\Delta\Gamma=\sum_{i=2}^{\infty}\Gamma^{(i)}[\phi]\,. \tag{2.17}$$

2.1. *Perturbative derivation of the graphical rule for $\Gamma[\phi]$ through the inversion method*

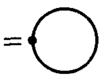
An explicit calculation up to the fourth order of λ is sketched, and based on the result the general rule for full order is deduced. Full justification is given in Subsec. 2.2. In Subsec. 2.1.1 the rule for $J^{(n)}$ is inferred by the inversion method. We see that $J^{(n)}$ is successively given as a functional of $J^{(0)}[\phi]$. Then in Subsec. 2.1.2 we obtain $\Gamma^{(n)}$ based on the $J^{(n)}$ vertex in two ways: by integrating the diagrams of $J^{(n)}$ or by starting from a closed formula for $\Delta\Gamma$. Since $J^{(n)}$ has already been given as a functional of $J^{(0)}[\phi]$ in Subsec. 2.1.1, the effective action $\Delta\Gamma$ is obtained as a functional of $J^{(0)}[\phi]$. Explicit rules for ΔJ and $\Delta\Gamma$ are given in Subsec. 2.1.3, in which their dependence on $J^{(0)}[\phi]$ is transparent. For this purpose an artificial bosonic field σ whose propagator is a functional of $J^{(0)}[\phi]$ is introduced.

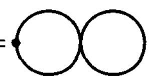
2.1.1. *Rule for* $J^{(n)}$

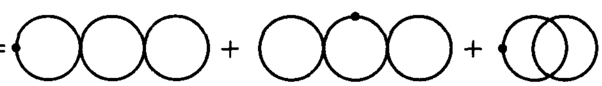
The original series of ϕ is first calculated as

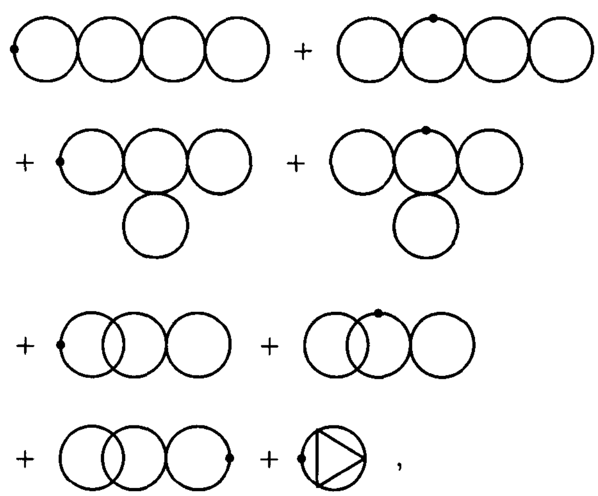
$$\phi = \phi^{(0)} + \phi^{(1)} + \phi^{(2)} + \phi^{(3)} + \phi^{(4)} + \cdots \tag{2.18}$$

through (2.9), regarding J as being of order unity with graphical representation as follows:

$\phi^{(0)}[J] =$  , (2.19)

$\phi^{(1)}[J] =$  , (2.20)

$\phi^{(2)}[J] =$  , (2.21)

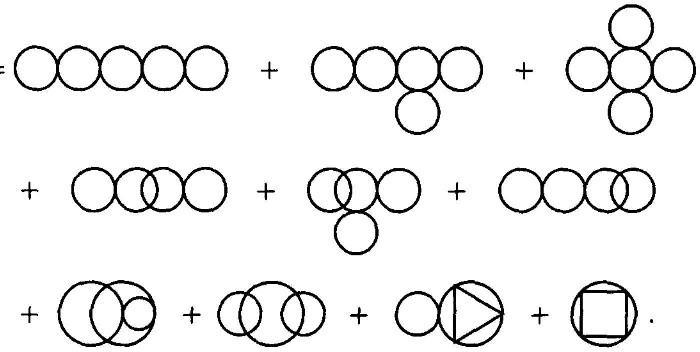
$\phi^{(3)}[J] =$  , (2.22)

$\phi^{(4)}[J] =$  + 30 diagrams. (2.23)

Here the dot • corresponds to an external point where two propagators meet and to the insertion of the operator $\varphi(x)^2$ which is effected by the derivative with respect to $J(x)$. Note here the relation $\frac{\partial G(y,z)}{\partial J(x)} = G(y,x)G(x,z)$. The propagator $G_J(x,y)$ and the factor $-\lambda$ are associated with a line and a four-point vertex respectively. (No factor is associated with a dot. For a detailed rule including the symmetry factor, see App. B.)

We mention here that the diagrams of $\phi^{(n)}$ are obtained by attaching a dot, in all possible ways, to one of the lines in the graphs of the nth order of W. For example, the 31 diagrams of $\phi^{(4)}$ are obtained through the fourth order of W:

$W^{(4)}[J] =$



(2.24)

Since the above diagrams of $\phi^{(i)}$ are all functionals of $J(x)$, which is contained in the propagator $G_J(x,y)$, we get $\phi(x)$ as a functional of J; $\phi = \phi[J]$. Assume that the relation $\phi = \phi[J]$ is *inverted* to give the relation $J = J[\phi]$ and this inversion is done perturbatively as in (1.2) regarding ϕ as an quantity independent of λ or the order $\lambda^0 = 1$. Then, as in the Introduction, we get the following formulae of the inversion method:

$$\phi = \phi^{(0)}[J^{(0)}]\,, \tag{2.25}$$

$$\phi^{(0)\prime} J^{(1)} + \phi^{(1)} = 0\,, \tag{2.26}$$

$$\phi^{(0)\prime} J^{(2)} + \tfrac{1}{2}\phi^{(0)\prime\prime}\big(J^{(1)}\big)^2 + \phi^{(1)\prime} J^{(1)} + \phi^{(2)} = 0\,, \tag{2.27}$$

$$\begin{aligned}&\phi^{(0)\prime} J^{(3)} + \phi^{(0)\prime\prime} J^{(1)} J^{(2)} + \frac{1}{3!}\phi^{(0)\prime\prime\prime}\big(J^{(1)}\big)^3 \\ &\quad + \phi^{(1)\prime} J^{(2)} + \frac{1}{2}\phi^{(1)\prime\prime}\big(J^{(1)}\big)^2 + \phi^{(2)\prime} J^{(1)} + \phi^{(3)} = 0\,,\end{aligned} \tag{2.28}$$

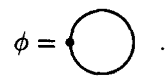
$$\begin{aligned}&\phi^{(0)\prime} J^{(4)} + \frac{1}{2}\phi^{(0)\prime\prime}\big(2J^{(1)}J^{(3)} + \big(J^{(2)}\big)^2\big) + \frac{1}{2}\phi^{(0)\prime\prime\prime}\big(J^{(1)}\big)^2 J^{(2)} \\ &\quad + \frac{1}{4!}\phi^{(0)\prime\prime\prime\prime}\big(J^{(1)}\big)^4 + \phi^{(1)\prime} J^{(3)} + \phi^{(1)\prime\prime} J^{(1)} J^{(2)} + \frac{1}{3!}\phi^{(1)\prime\prime\prime}\big(J^{(1)}\big)^3 \\ &\quad + \phi^{(2)\prime} J^{(2)} + \frac{1}{2}\phi^{(2)\prime\prime}\big(J^{(1)}\big)^2 + \phi^{(3)\prime} J^{(1)} + \phi^{(4)} = 0\,.\end{aligned} \tag{2.29}$$

Here we have employed a concise notation. If we explicitly write (2.27), for example, it has the form

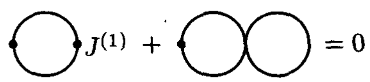
$$\begin{aligned}&\int d^4x \frac{\delta\phi^{(0)}[J^{(0)}]}{\delta J^{(0)}(x)} J^{(2)}(x) + \frac{1}{2}\int d^4x d^4y \frac{\delta\phi^{(0)}[J^{(0)}]}{\delta J^{(0)}(x)\delta J^{(0)}(y)} J^{(1)}(x) J^{(1)}(y) \\ &\quad + \int d^4x \frac{\delta\phi^{(1)}[J^{(0)}]}{\delta J^{(0)}(x)} J^{(1)}(x) + \phi^{(2)}[J^{(0)}] = 0\,.\end{aligned} \tag{2.30}$$

We emphasize here that all $\phi^{(i)}$ $(i = 0, 1, 2, \ldots)$ and their derivatives in (2.25)–(2.29) are evaluated at $J = J^{(0)}[\phi]$, defined implicitly by (2.25). So Eqs. (2.26)–(2.29) successively give the functional dependence of $J^{(1)}$ to $J^{(4)}$ on ϕ *through* $J^{(0)}[\phi]$.

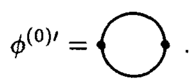
Let us discuss the graphical expressions of (2.25)–(2.29). Note here that the propagator in the following graphs is $G^{(0)} = \frac{1}{\Box + m^2 - J^{(0)}[\phi]}$ instead of G_J. Then, from (2.19), Eq. (2.25) is expressed as


(2.31)

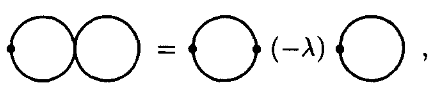
Here and hereafter the dot represents a derivative not by J but by $J^{(0)}$. Notice also that the meanings of the graphs on the right hand sides of (2.19) and (2.31) are different because the line or the propagator in them is not the same: G_J for (2.19) and $G^{(0)}$ for (2.31). Thus (2.31) reduces to (2.16). It is stressed here that $J^{(0)}[\phi]$ is defined through (2.16) or (2.31) although its dependence on ϕ is only implicit. By the use of (2.19) and (2.20), Eq. (2.26) is also expressed as follows:


(2.32)

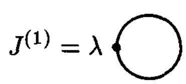
Here we have used the relation


(2.33)

Noting that a four-point vertex makes a contribution $-\lambda$ so that


(2.34)

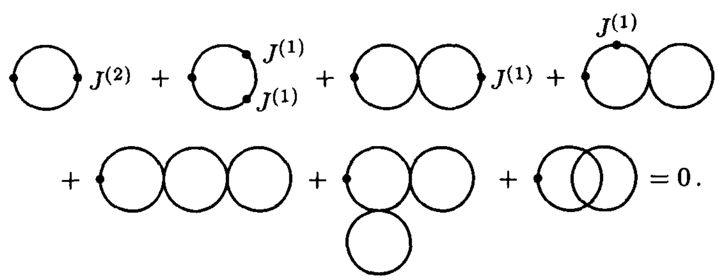
we get from (2.32)


(2.35)

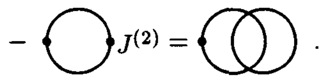
or

$$J^{(1)} = \lambda\phi = \lambda \frac{1}{2i} \mathrm{Tr} \frac{1}{\Box + m^2 - J^{(0)}[\phi]} \,. \tag{2.36}$$

Thus $J^{(1)}$ is given by $J^{(0)}[\phi]$. Consider next the graphical expression of (2.27) obtained through (2.19)–(2.21).

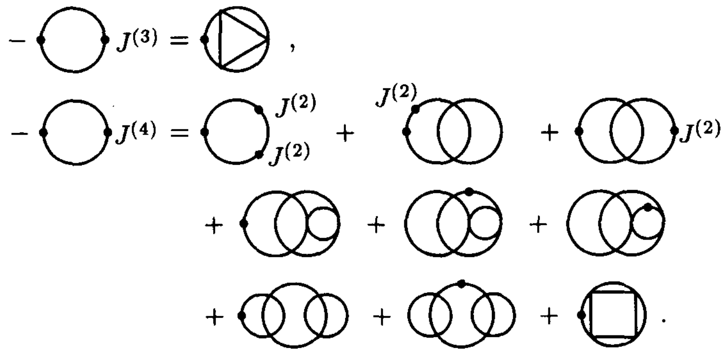

(2.37)

We see that the second, fourth and sixth graphs on the left hand side are summed up to zero after replacing $J^{(1)}$ by the right hand side of (2.35) by explicitly taking symmetry factors into account of course — see App. B. A similar cancellation of the third and fifth graphs on the left hand side of (2.37) occurs, ending up with



(2.38)

The graphs of (2.28) and (2.29) are also obtained through (2.19)–(2.24). These expressions originally consisted of many terms, but due to a similar cancellation mechanism they reduce to



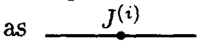
(2.39)

(2.40)

These simple results lead us to the following proposition to be justified later. Before we present the proposition it is convenient to introduce the terms 1VI and 1VR. The 1VI (one-vertex-irreducible) graph is a connected graph in which removal of any one of the four-point vertices does not lead to two separate graphs. The 1VR (one-vertex-reducible) vertex is defined as a four-point vertex in a connected diagram whose deletion results in a separation of the graph. The 1VI graph can also be defined as the connected graph without any 1VR vertex while the 1VR graph is a graph in which at least one 1VR vertex is present. By definition *a graph which does not have any four-point vertex is not 1VR but 1VI* although the trivial skeleton (Tr ln term) is not 1VR or 1VI. Namely, all the graphs are classified into three categories: 1VI, 1VR and the trivial skeleton. For later convenience we introduce the skeleton. Both the 1VI graph and the trivial skeleton are called the skeleton. In other words, the whole class of the skeleton comprises all the 1VI graphs plus the trivial skeleton. With this terminology we see that all the 1VR graphs in (2.37) disappear to result in (2.38) after all the $J^{(1)}$'s are replaced by the right hand side of (2.35). Thus we can deduce the following proposition.

Proposition A1. After the replacement of $J^{(1)}$ by its graphical expression of the right hand side of (2.35), all the 1VR graphs originally appearing in the inversion formula of the nth order with $n \geq 2$ [(2.27)–(2.29) and higher order relations] cancel out. In other words, only the 1VI graphs with correct weight remain in the inversion formulae.

Note here that 1VI cannot be replaced by 2PI (two-particle-irreducible) as in the case of the effective action for the *nonlocal* operator $\varphi(x)\varphi(y)$.[2,4,6] This is clear from the second and third (1VI) graphs from the last on the left hand side of (2.40), which are 2PR (two-particle-reducible).

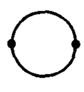
We also note a very convenient way to express the *original* graphs of the inversion formulae (2.25)–(2.29) and higher order relations such as (2.31), (2.32) and (2.37) in which graphs $J^{(1)}$'s still remain [without replacing them by the right hand side of (2.35)]. Let us turn our attention to (2.10), where graphs are built with propagators $G^{(0)}$ and four-point vertices $-\lambda$ and *pseudovertices* of order λ^i with $i \geq 1$, which is denoted as  $J^{(i)}$. We have called the two-point vertex originating from $J^{(n)}\varphi^2$ a *pseudovertex*, since it has nothing to do with the definition of 1VI. The term 1VI is defined as one-vertex-irreducible with respect to the four-point vertex. Then the graphs of the inversion formula are obtained as follows. *If one writes down the nth order of (2.10) considering ϕ and $G^{(0)}$ (namely, $J^{(0)}$) as being of order $\lambda^0 = 1$, one obtains the inversion formula of order n in the graphical form.* For example, the zeroth order of (2.10) is

$$\phi = \left\langle \frac{1}{2}\varphi^2 \right\rangle_{G^{(0)}}, \tag{2.41}$$

which is equivalent to (2.31), and the first order is

$$0 = \left\langle \frac{1}{2}\varphi^2 \left(-\frac{i\lambda}{4!}\varphi^4 + \frac{i}{2}J^{(1)}\varphi^2 \right) \right\rangle_{G^{(0)}}, \tag{2.42}$$

which is (2.32). Furthermore the second order of (2.10) reduces to (2.37). Here it is convenient to introduce the *self-contraction* of the pseudovertex (Fig. 1) and the four-point vertex (Fig. 2). Since the relation $\frac{\delta G^{(0)}}{\delta J^{(0)}} = G^{(0)}G^{(0)}$ holds, the quantity



$$J^{(n)} = \left\langle \frac{1}{2}\varphi^2 \cdot \frac{i}{2}J^{(n)}\varphi^2 \right\rangle_{G^{(0)}} = \frac{1}{2i}\frac{1}{\Box + m^2 - J^{(0)}}\frac{1}{\Box + m^2 - J^{(0)}}J^{(n)}$$

$$= \frac{\delta}{\delta J^{(0)}}$$

$$J^{(n)} \tag{2.43}$$

is called (the nth order of) the *derivative of the self-contraction* in the following. Then, as we have from (2.10)

$$0 = n\text{th order of } \left\langle \frac{1}{2}\varphi^2 e^{-\frac{i\lambda}{4!}\varphi^4 + \frac{i}{2}(J^{(1)} + J^{(2)} + \cdots)\varphi^2} \right\rangle_{G^{(0)}}, \tag{2.44}$$

Fig. 1. The self-contraction of the pseudovertex.

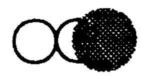
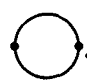


Fig. 2. The self-contraction of the four-point-vertex.

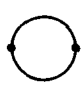
for $n \geq 1$ we get the formula



$$- \bigcirc J^{(n)} = n\text{th order of } \left\langle \frac{1}{2}\varphi^2 e^{-\frac{i\lambda}{4!}\varphi^4+\frac{i}{2}(J^{(1)}+J^{(2)}+\cdots)\varphi^2} \right\rangle_{G^{(0)}}^{\text{ndself}} , \tag{2.45}$$

where ndself (no derivative of the self-contraction) implies that the derivative of the self-contraction is moved on the left hand side. Since (2.45) is the original inversion formulae of the nth order, Proposition A1 implies that in (2.45) all the contributions from the $J^{(1)}$ vertices should be eliminated if only 1VI graphs are kept. Thus Proposition A1$'$ follows:

Proposition A1$'$. $J^{(n)}[\phi]$ $(n \geq 2)$ is successively obtained *as a functional of* $J^{(0)}[\phi]$ through



$$- \bigcirc J^{(n)} = n\text{th order of } \left\langle \frac{1}{2}\varphi^2 e^{-\frac{i\lambda}{4!}\varphi^4+\frac{i}{2}(J^{(2)}+J^{(3)}+\cdots)\varphi^2} \right\rangle_{G^{(0)}}^{\text{1VI/ndself}} , \tag{2.46}$$

i.e. the sum of all the connected 1VI/ndself diagrams with one external point built with the four-point vertex of $-\lambda$, the two-point pseudovertex $J^{(i)}$ $(i \geq 2)$ and the propagator $G^{(0)}$.

The restriction 1VI/ndself implies that the derivatives by $J^{(0)}$ of the self-contracted diagram are excluded and, at the same time, only the 1VI graphs should be kept. This proposition is directly proved in the next subsection by using the summing-up rule.[16]

We can *directly* get (2.38)–(2.40) from Proposition A1$'$ due to the 1VI/ndself restriction [through the procedure similar to the one given in (2.41) or (2.42), etc.]. We notice that the right hand side of (2.46) contains $J^{(i)}$'s with $i < n$ (strictly speaking $i \leq n-2$). Hence we successively obtain $J^{(i)}$ $(i \geq 2)$ as a functional of $J^{(0)}$. For example, $J^{(4)}$ is expressed by $J^{(0)}$ if one inserts the $J^{(2)}$ vertex given in (2.38) into (2.40). In this way $J^{(i)}$ can be successively given as a functional of $J^{(0)}[\phi]$ [without using $J^{(j)}$ $(j < i, i \neq 0)$].

2.1.2. *Rule for $\Delta\Gamma$ by use of the pseudovertex $J^{(n)}$ with $n \geq 2$*

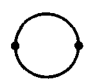
Now we turn our attention to the effective action $\Gamma[\phi]$ itself. We notice that $\Gamma^{(n)}[\phi]$ satisfies

$$\frac{\delta\Gamma^{(n)}[\phi]}{\delta\phi(x)} = -J^{(n)}(x)\,. \tag{2.47}$$

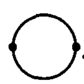
Then for $n = 0$ we get (2.14), because by differentiating the right hand side of (2.14) with respect to ϕ we obtain $-J^{(0)}$ through (2.16). $\Gamma^{(1)}$ is also easily obtained by integration of (2.36) so that we have (2.15). To derive $\Gamma^{(n)}$ for higher n, it is convenient to note the fact that

$$-J^{(n)} = \frac{\delta\Gamma^{(n)}[\phi]}{\delta\phi} = \frac{\delta J^{(0)}[\phi]}{\delta\phi}\frac{\delta\Gamma^{(n)}[\phi]}{\delta J^{(0)}} \tag{2.48}$$

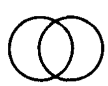
and that

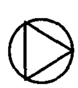
$$\frac{\delta\phi}{\delta J^{(0)}} = -D^{-1} =$$  . (2.49)

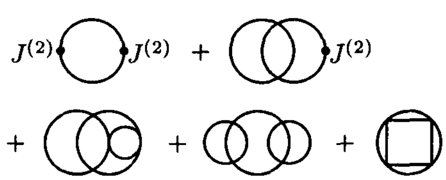
The quantity of the last equation is a kind of propagator for the composite operator $\varphi(x)^2$ and is called the *composite propagator* $[\langle\varphi(x)^2\varphi(y)^2\rangle]$. D is called the inverse composite propagator in what follows. Thus we get from (2.48)

$-$  $J^{(n)} = \dfrac{\delta\Gamma^{(n)}}{\delta J^{(0)}}\,.$ (2.50)

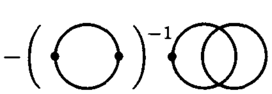
Therefore the right hand side of (2.46) is just $\frac{\delta\Gamma^{(n)}}{\delta J^{(0)}}$. Keeping (2.50) in mind and by integrating (2.38)–(2.40) we arrive at

$\Gamma^{(2)} =$  , (2.51)

$\Gamma^{(3)} =$  , (2.52)

$\Gamma^{(4)} =$  , (2.53)

with

$J^{(2)} = -$  . (2.54)

Note that the symmetry factors play an important role in the integration (see App. B). Note also that the first factor on the right hand side of (2.54) corresponds to the *amputation* of the composite propagator (2.49).

In fact we can derive (2.51)–(2.53) and higher order relations more easily. To this end we introduce a closed form of functional representation of $\Delta\Gamma[\phi]$. We first write down the following equation, which is clear from (2.1), (2.2) and (2.11):

$$e^{i\Gamma[\phi]} = \int \mathcal{D}\varphi e^{i\left[-\frac{1}{2}\varphi(\Box+m^2)\varphi-\frac{1}{4!}\lambda\varphi^4+\frac{1}{2}J\varphi^2-J\phi\right]}, \tag{2.55}$$

where J is expressed by ϕ. Noting that

$$J = J^{(0)}[\phi] + \lambda\phi + \Delta J[\phi] \tag{2.56}$$

with

$$\Delta J = J^{(2)} + J^{(3)} + \cdots = -\frac{\delta\Delta\Gamma}{\delta\phi} \tag{2.57}$$

and that, apart from the irrelevant constant factor,

$$\int \mathcal{D}\varphi e^{-i\frac{1}{2}\varphi(\Box+m^2-J^{(0)})\varphi} = e^{-\frac{1}{2}\mathrm{Tr}\,\ln(\Box+m^2-J^{(0)})}, \tag{2.58}$$

we get

$$e^{i\Gamma[\phi]} = e^{i\left[-J^{(0)}\phi-\frac{1}{2i}\mathrm{Tr}\,\ln\left(\Box+m^2-J^{(0)}\right)-\frac{\lambda}{2}\phi^2\right]} \times \frac{\int \mathcal{D}\varphi e^{i\left[-\frac{1}{2}\varphi\left(\Box+m^2-J^{(0)}\right)\varphi-\frac{1}{4!}\lambda\varphi^4+\frac{1}{2}\left(J-J^{(0)}\right)\varphi^2-\left(J-J^{(0)}-\frac{\lambda}{2}\phi\right)\phi\right]}}{\int \mathcal{D}\varphi e^{-i\frac{1}{2}\varphi(\Box+m^2-J^{(0)})\varphi}}. \tag{2.59}$$

In this way a closed formula for $\Delta\Gamma$ is obtained:

$$e^{i\Delta\Gamma[\phi]} = \frac{\int \mathcal{D}\varphi e^{i\left[-\frac{1}{2}\varphi\left(\Box+m^2-J^{(0)}\right)\varphi+\left\{-\frac{1}{4!}\lambda\varphi^4+\frac{\lambda}{2}\phi\varphi^2-\frac{\lambda}{2}\phi^2\right\}-\frac{\delta\Delta\Gamma}{\delta\phi}\left(\frac{\varphi^2}{2}-\phi\right)\right]}}{\int \mathcal{D}\varphi e^{-i\frac{1}{2}\varphi(\Box+m^2-J^{(0)})\varphi}}. \tag{2.60}$$

This equation indicates that $\Delta\Gamma$ can be calculated perturbatively by using $G^{(0)} = (\Box + m^2 - J^{(0)})^{-1}$ as propagators. The role of the *additional vertices* $\frac{\lambda}{2}\phi\varphi^2 - \frac{\lambda}{2}\phi^2$ and $\frac{\delta\Delta\Gamma}{\delta\phi}\phi$ are merely to suppress the self-contractions. In other words the graphs having the structure of Figs. 1 and 2 disappear. To see this let us take one specific $-\frac{\lambda}{4!}\varphi^4$ vertex. Each of four φ's of the vertex is to be contracted with the other φ. There are three possible ways to make such contractions:

$$-\frac{\lambda}{4!}\varphi^4 \Rightarrow -\frac{\lambda}{4!}:\varphi^4: -\frac{\lambda}{4!}\frac{4\cdot 3}{2}\widehat{\varphi\varphi}:\varphi^2: -\frac{\lambda}{4!}\frac{4\cdot 3}{2\cdot 2}\widehat{\varphi\varphi}\widehat{\varphi\varphi}, \tag{2.61}$$

where the normal ordering $:\varphi^n:$ means that each of the n φ's is to be contracted with φ contained in a vertex different from the one we are taking. Note here that the contraction within a single vertex (self-contraction) is given by

$$\widehat{\varphi\varphi} = \frac{1}{i}\left(\frac{1}{\Box + m^2 - J^{(0)}}\right)_{xx} = 2\phi\,. \tag{2.62}$$

We can write in a similar manner

$$\frac{\lambda}{2}\phi\varphi^2 \Rightarrow \frac{\lambda}{2}\phi : \varphi^2 : + \frac{\lambda}{2}\phi\widehat{\varphi\varphi}\,. \tag{2.63}$$

Then the contractions of the set that appeared in (2.60) become

$$-\frac{\lambda}{4!}\varphi^4 + \frac{\lambda}{2}\phi\varphi^2 - \frac{\lambda}{2}\phi^2 \Rightarrow -\frac{\lambda}{4!} : \varphi^4 : , \tag{2.64}$$

which is clear from (2.61)–(2.63). In the same way $\frac{\delta\Delta\Gamma}{\delta\phi}\left(\frac{\varphi^2}{2} - \phi\right)$ reduces to

$$\frac{\delta\Delta\Gamma}{\delta\phi}\left(\frac{1}{2}(:\varphi^2: + \widehat{\varphi\varphi}) - \phi\right) = \frac{\delta\Delta\Gamma}{\delta\phi}\frac{1}{2} : \varphi^2 : . \tag{2.65}$$

In this way we get a simple formula for $\Delta\Gamma[\phi]$:

$$\Delta\Gamma = \frac{1}{i}\left\langle e^{-\frac{i\lambda}{4!}\varphi^4 - \frac{i}{2}\frac{\delta\Delta\Gamma}{\delta\phi}\varphi^2}\right\rangle^{\mathrm{nself}}_{G^{(0)}} = \frac{1}{i}\left\langle e^{-\frac{i\lambda}{4!}\varphi^4 + \frac{i}{2}(J^{(2)} + J^{(3)} + \cdots)\varphi^2}\right\rangle^{\mathrm{nself}}_{G^{(0)}}, \tag{2.66}$$

where the superscript nself implies that we have to keep all possible connected Wick contractions using the propagator $G^{(0)} = (\Box + m^2 - J^{(0)})^{-1}$ *excluding self-contractions* of both the four-point vertex and the pseudovertices.

From this formula we can successively derive $\Gamma^{(n)}$ $(n \geq 2)$ more easily than in the previous method in which we started from the algebraic inversion formula to obtain $J^{(n)}$ first and then $\Gamma^{(n)}$ through integration. This is seen as follows. First notice that (2.66) actually starts from λ^2 because the first order of the right hand side of (2.66), which is $\frac{1}{i}\left\langle -\frac{i\lambda}{4!}\varphi^4\right\rangle^{\mathrm{nself}}_{G^{(0)}}$, becomes zero due to the nself restriction. Since $\Gamma^{(2)}$ is of order λ^2, we get

$$\Gamma^{(2)} = \frac{1}{i}\left\langle \frac{1}{2}\left(-\frac{i\lambda}{4!}\varphi^4\right)^2 - \frac{i}{2}J^{(2)}\varphi^2\right\rangle^{\mathrm{nself}}_{G^{(0)}}. \tag{2.67}$$

The second term on the right hand side makes no contribution to $\Gamma^{(2)}$, again due to the nself condition, thus leading to (2.51). In the same way $\Gamma^{(3)}$ is calculated from the expression

$$\Gamma^{(3)} = \frac{1}{i}\left\langle \frac{1}{3!}\left(\frac{-i\lambda}{4!}\varphi^4\right)^3 - i\frac{\lambda}{4!}\varphi^4\left(-\frac{i}{2}J^{(2)}\varphi^2\right) - \frac{i}{2}J^{(3)}\varphi^2\right\rangle^{\mathrm{nself}}_{G^{(0)}}, \tag{2.68}$$

and we get (2.52).

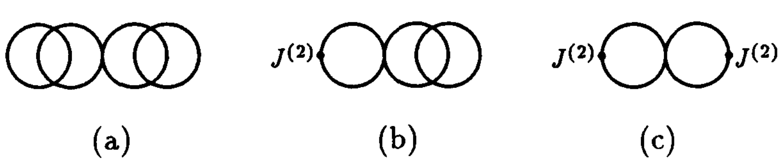


Fig. 3. An example of 1VR graph that is canceled in $\Gamma^{(5)}$.

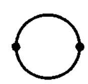
This course of study can be continued (up to the desired order) to get (2.53) and so on. In (2.66) we do not yet have the 1VI restriction explicitly, but we can see that, due to the additional vertex $-\frac{i}{2}\frac{\delta\Delta\Gamma}{\delta\phi}\varphi^2 = -\frac{i}{2}\Delta J\varphi^2$, all the 1VR structures in the diagrammatic expression of (2.66) exactly cancel out. For example, the 1VR graph of Fig. 3(a) which appeared in (2.66) for $n = 5$ is canceled by those of Figs. 3(b) and 3(c), which are supplied by the pseudovertex $-\frac{i}{2}\Delta J\varphi^2$. Thereby a practical formula for $\Delta\Gamma[\phi]$ is obtained:

Proposition A2. $\Delta\Gamma$ is given by the rule

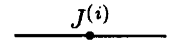
$$\Delta\Gamma = \frac{1}{i}\left\langle e^{-\frac{i\lambda}{4!}\varphi^4+\frac{i}{2}(J^{(2)}+J^{(3)}+\cdots)\varphi^2}\right\rangle_{G^{(0)}}^{\rm 1VI/nself}, \tag{2.69}$$

i.e. the sum of all the connected 1VI/nself vacuum diagram built with four-point vertices of $-\lambda$, two-point pseudovertices of $J^{(i)}$ $(i \geq 2)$ and propagators $G^{(0)}$.

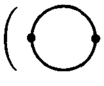
The condition 1VI/nself implies that only the connected Wick contraction corresponding to the 1VI graph needs to be considered and, at the same time, that the self-contractions of the pseudovertex of Fig. 1 are excluded. The self-contractions of the four-point vertex of Fig. 2 are automatically excluded by the restriction of 1VI. Corresponding to the relation (2.50) or

$-$  $\Delta J = \dfrac{\delta\Delta\Gamma}{\delta J^{(0)}}\,,$ (2.70)

the ndself restriction in (2.46) is changed to nself in (2.69). Proposition A1′ is the derivative form of Proposition A2. Proposition A2 is clearly equivalent to the following Proposition A2′ and is justified rigorously in the next subsection.

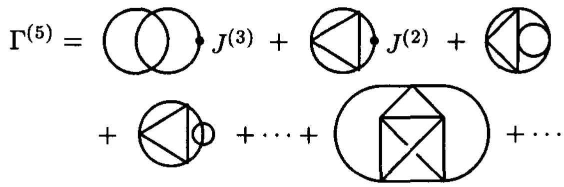
Proposition A2′. $\Gamma^{(n)}$ $(n \geq 2)$ is the sum of all possible nth order 1VI/nself diagrams constructed out of the four-point vertex of order λ and the *pseudovertices* of the order λ^i $(2 \leq i < n-2)$, which are denoted by  $J^{(i)}$, and the propagator $G^{(0)} = (\Box + m^2 - J^{(0)})^{-1}$.

We put stress on the fact that Proposition A2 or A2′ makes it possible to write down $\Gamma^{(n)}$ $(n \geq 2)$ successively with its ϕ dependence coming only through $J^{(0)}[\phi]$, although the rule contains $J^{(2)}$, $J^{(3)}$, $J^{(4)}, \ldots$. This is because the graphs of $\Gamma^{(n)}$ contain $J^{(i)}$ with $i \leq n-2$ while the graphical rule for these $J^{(i)}$ in terms of $G^{(0)}$ propagators is already known in (2.46) or through $\Gamma^{(i)}$ by the relation (2.48):

$J^{(i)} = -\Big($  $\Big)^{-1} \dfrac{\delta\Gamma^{(i)}}{\delta J^{(0)}}\,.$ (2.71)

Combined with the fact that $\Gamma^{(0)}$ and $\Gamma^{(1)}$ are also given only through $J^{(0)}$, which is clear from (2.14) and (2.15) with (2.16), Γ itself is given by $J^{(0)}$.

From Proposition A2 or A2′ we can directly obtain (2.51)–(2.53) and



(2.72)

and so on. The directness comes from the 1VI restriction.

Now it is convenient to introduce the whole class of the 1VI vacuum graph $\mathcal{K}[A]$;

$$\mathcal{K}[A] = \langle e^{-\frac{i\lambda}{4!}\varphi^4} \rangle_A^{1VI}, \tag{2.73}$$

where the propagator used in the diagram is A. Note that the trivial skeleton $-\frac{1}{2i}\mathrm{Tr}\ln A^{-1}$ is not contained in $\mathcal{K}[A]$ by the definition (2.7). Thus this quantity is described as the whole class of the vacuum skeleton minus the trivial skeleton. The whole class of the vacuum skeleton is given by

$$\bar{\mathcal{K}}[A] = \mathcal{K}[A] - \frac{1}{2i}\mathrm{Tr}\ln A^{-1} = \int \mathcal{D}\varphi e^{-\frac{1}{2}\varphi A^{-1}\varphi - \frac{i\lambda}{4!}\varphi^4}\Big|_{\text{excl 1VR}}, \tag{2.74}$$

where excl 1VR implies that the 1VR graphs are excluded or that only the 1VI graph and the trivial skeleton are kept.

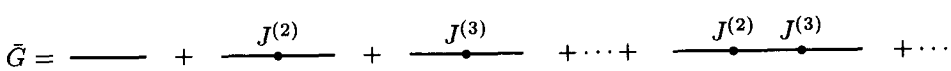
In (2.53), (2.72) and in the graphs of $\Gamma^{(n)}$ with higher n obtained by Proposition A2′, we see that $\Gamma^{(n)}$ is the sum of all the 1VI vacuum diagrams built with the four-point vertex and the decorated $G^{(0)}$ propagator. The decoration is done by $J^{(n)}$ $(n \geq 2)$ pseudovertices which are inserted into the $G^{(0)}$ propagators in all possible ways. We see also that $-\frac{1}{2i}\mathrm{Tr}\ln[G^{(0)}]^{-1}$ and the self-contractions of the pseudovertex $J^{(i)}$ with $i \geq 2$ are not included in $\Delta\Gamma$. Thereby we arrive at Proposition A2″:

Proposition A2″. $\Delta\Gamma[\phi]$ is given by $\mathcal{K}[\bar{G}] - \frac{1}{2i}\mathrm{Tr}\ln[\bar{G}]^{-1} - \Delta\mathcal{K}_{\mathrm{tr}} = \bar{\mathcal{K}}[\bar{G}] - \Delta\mathcal{K}_{\mathrm{tr}}$, where

$$\bar{G} = \left(\Box + m^2 - J^{(0)} - J^{(2)} - J^{(3)} - \cdots\right)^{-1} \tag{2.75}$$

$$= \left(\Box + m^2 + \lambda\phi - J[\phi]\right)^{-1} \tag{2.76}$$

or, with the line representing the propagator $G^{(0)}$,



(2.77)

and

$$\Delta\mathcal{K}_{\mathrm{tr}} = -\frac{1}{2i}\mathrm{Tr}\ln[G^{(0)}]^{-1} + \phi\left(J - J^{(0)} - J^{(1)}\right). \tag{2.78}$$

In other words, $\Gamma = \Gamma^{(0)} + \Gamma^{(1)} + \Delta\Gamma$ is given by

$$\begin{aligned}\Gamma[\phi] &= -\phi J[\phi] + \frac{\lambda}{2}\phi^2 - \frac{1}{2i}\mathrm{Tr}\ln\left(\Box + m^2 - J[\phi] + \lambda\phi\right) + \mathcal{K}[\bar{G}] \\ &= -\phi J[\phi] + \frac{\lambda}{2}\phi^2 + \bar{\mathcal{K}}[\bar{G}]\,. \end{aligned} \tag{2.79}$$

The quantity $-\frac{1}{2i}\mathrm{Tr}\ln[\bar{G}]^{-1} - \Delta\mathcal{K}_{\mathrm{tr}}$ is a Tr ln of a decorated propagator specified as follows. The decoration is made by $J^{(0)}$, $J^{(2)}$, $J^{(3)}, \ldots$ ($J^{(1)} = \lambda\phi$ is not included) but the decoration only by $J^{(0)}$'s [the first term on the right hand side of (2.78)] and the decoration by one single $J^{(2)}, J^{(3)}, \ldots$ [the last term of (2.78), which is a summation of the self-contracted diagrams of Fig. 1 with $i \geq 2$] are excluded. Proposition A2$''$ will be justified in the next subsection precisely.

Although the appearance of the term $-J\phi$ in (2.79) seems to be somewhat curious, it is not actually so. Differentiating (2.79) with respect to ϕ by noting (2.76) and (2.12), we get

$$\left(\phi + \frac{\delta\bar{\mathcal{K}}}{\delta\bar{G}^{-1}}\right)\left(-\frac{\delta J}{\delta\phi} + \lambda\right) = 0\,. \tag{2.80}$$

The second term is not zero because $\frac{\delta J}{\delta\phi}$ contains various orders of λ. Thus we get

$$\phi = -\frac{\delta\bar{\mathcal{K}}}{\delta\bar{G}^{-1}}\,, \tag{2.81}$$

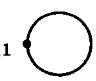
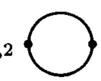
which is consistent with (2.8) or $\phi = \frac{\delta W}{\delta J}$ in the following sense. If one uses the relation

$$W = \Gamma + J\phi = \frac{\lambda}{2}\phi^2 + \bar{\mathcal{K}}[\bar{G}] \tag{2.82}$$

[obtained from (2.79)] on the right hand side of $\phi = \frac{\delta W}{\delta J}$ and then uses (2.81), one gets the left hand side of this equation, i.e. ϕ.

2.1.3. *Rule for* $\Delta\Gamma$ *in terms of* $J^{(0)}[\phi]$

From Propositions A2 and A1$'$ we can deduce another graphical rule for $\Delta\Gamma$ and ΔJ in which all the ϕ dependence is explicitly through $J^{(0)}[\phi]$. We arrive at the new rule by using (2.71) and by noting that $\frac{\delta\Gamma^{(i)}}{\delta J^{(0)}}$ is given by the right hand side of (2.46) (in addition to the facts stated just above Proposition A2$''$). To state the new rule we introduce the i-vertex ($i = 0, 1, 2, \ldots$), which is defined as

$$v_i(x_1, \ldots, x_i) \equiv \frac{1}{i!}\frac{\delta^i\bar{\mathcal{K}}[G^{(0)}]}{\delta J^{(0)}(x_1)\cdots\delta J^{(0)}(x_i)} - \delta_{i,1}$$  $$- \delta_{i,2}$$  , (2.83)

where $\delta_{i,j}$ is the Kronecker delta.

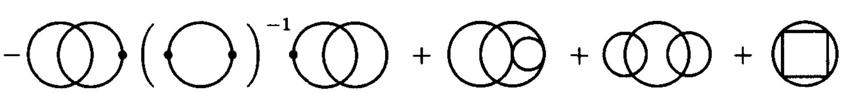
Now the final rule is given by the following statement where *the graphs are built with the inverse composite propagator* $D(x,y)$ *and the vertices* $v_i(x_1,\ldots,x_i)$ $(i=0,1,2,\ldots)$.

Proposition A3. $\Delta\Gamma$ and ΔJ are given by the following rules:

$$\Delta\Gamma = \sum \big[\text{all the connected } \textit{tree} \text{ diagrams with all the pairs } (x_l, x_m) \text{ of the arguments of } v_i\text{'s connected by } D \text{ propagators (vacuum graph)}\big]\,, \tag{2.84}$$

$$\Delta J(x) = \int d^4y D_{xy} \times \sum \big[\text{all the connected } \textit{tree} \text{ diagrams with one of the arguments of one of the } v_i\text{'s being the point } y \text{ (graph with an external point)}\big]\,. \tag{2.85}$$

The tree graph in terms of D propagator is the graph in which all the D propagators are *articulate.* Here the D propagator in a connected graph is called *articulate* if removal of it leads to a separation of the graph. Note that $D(x,y)$ lines never make a loop because $D(x,y)$ comes from $J^{(i)}$ with $i \geq 2$ [see (2.71)]. Proposition A3 is understood through examples. For instance, $\Gamma^{(4)}$ in (2.53) or

$\Gamma^{(4)} = -$  (2.86)

can be written as

$$\Gamma^{(4)} = \text{ fourth order of } \left(\frac{1}{2}v_1 D v_1 + v_0\right)\,. \tag{2.87}$$

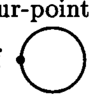
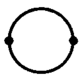
In (2.86) $J^{(0)}[\phi]$ dependence is evident because there is no $J^{(i)}$ pseudovertex $(i \geq 2)$. All the ϕ dependence is through $J^{(0)}$ contained in $G^{(0)}$ (and D). The sum of the first two terms of (2.53) exactly coincide with the first term of (2.86) with correct weight after substitution of (2.38) or (2.54).

Proposition A3 can be expressed as follows. For this purpose we introduce a σ field whose propagator is D. (The σ field looks like the auxiliary field but has nothing to do with it.) Then $\Delta\Gamma$ is given by

$$\begin{aligned} \Delta\Gamma &= \frac{1}{i}\left.\frac{\int \mathcal{D}\varphi\mathcal{D}\sigma e^{iS_0} e^{iS_{\rm int}}}{\int \mathcal{D}\varphi\mathcal{D}\sigma e^{iS_0}}\right|_{\rm conn/tree/1VI/excl}\,, \\ S_0 &= -\frac{1}{2}\varphi\big[G^{(0)}\big]^{-1}\varphi - \frac{1}{2}\sigma D^{-1}\sigma\,, \\ S_{\rm int} &= -\frac{\lambda}{4!}\varphi^4 + \frac{1}{2}\sigma\varphi^2\,, \end{aligned} \tag{2.88}$$

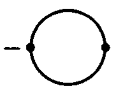
with

$D = -\frac{\delta J^{(0)}}{\delta \phi} = -\Big($  $\Big)^{-1}$. (2.89)

Here the subscript conn/tree/1VI/excl implies that only *connected* graphs which are *tree* graphs in terms of the D propagator and also *1VI* in terms of the four-point vertex have to be considered and, at the same time, that the substructure of  and  has to be excluded. Hence with the compact and self-evident notation, Proposition A3 is rewritten in the following form:

Proposition A3′. $\Delta\Gamma$ and ΔJ are given by the following formulae:

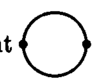
$$\Delta\Gamma = \frac{1}{i}\langle e^{-\frac{i\lambda}{4!}\varphi^4+\frac{i}{2}\sigma\varphi^2}\rangle_{G^{(0)},D}^{\text{tree/1VI/excl}} , \tag{2.90}$$



$$-\ \text{(diagram)}\ \Delta J = \left\langle \frac{1}{2}\varphi^2 e^{-\frac{i\lambda}{4!}\varphi^4+\frac{i}{2}\sigma\varphi^2}\right\rangle_{G^{(0)},D}^{\text{tree/1VI/excl}} , \tag{2.91}$$

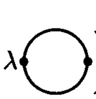
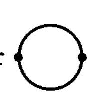
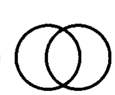
where the connected graphs with tree/1VI/excl restriction are constructed by three-point ($\sigma\phi^2$) and four-point ($\lambda\phi^4$) vertices and propagators $G^{(0)}$ of the φ field and D of the σ field.

Recall that both $G^{(0)}$ and D are functionals of $J^{(0)}$. Proposition A3′ is easily understood from the rule (2.69) with (2.46), but a rigorous proof is presented in Subsec. 2.2. Notice that this rule does not contain the $J^{(i)}$ pseudovertex unlike the previous rules, but instead D is represented by the propagator of the artificially introduced σ field. From (2.90), the quantity $\Gamma^{(4)}$, for example, can be *directly* obtained as (2.86) above.

Finally we note that a certain infinite series of the graphs appearing in $\Delta\Gamma[\phi]$ can be conveniently summed up. The series Γ_{ch} is the sum of all the possible closed chains constructed out of the unit element  or

$\Gamma_{\text{ch}} =$  $+\cdots .$ (2.92)

This series is summed up to give

$\Gamma_{\text{ch}} = -\frac{1}{2}\Big[\text{Tr}\ln\big(1-\lambda$  $\big) + \lambda\text{Tr}$  $+ (3!-2)$  $\Big]$. (2.93)

2.2. *Formal justification of propositions*

In this subsection we directly prove Propositions A2″, A1′ and A3′, leading to the full proof of all the propositions in the previous subsection.

Proposition A2″ or (2.79) is proved easily by analyzing the graphic expression of $W[J]$ rather than that of $\Gamma[\phi]$. It is based on a similar topological proof given in Ref. 2. If one writes the graph rule of $W[J]$ using $G_J = (\Box + m^2 - J)^{-1}$ as the propagator [the rule (2.4)], the whole dependence of $W[J]$ on J is through the propagator $G_J = (\Box + m^2 - J)^{-1}$. The contribution of all the 1VI graphs appearing in $W[J]$ can be written as $\mathcal{K}[G_J]$, the vacuum skeleton minus the trivial skeleton $-\mathrm{Tr}\ln G_J^{-1}$ [see (2.73)]. Then all the graphs of $W[J]$ seem to be generated by replacing G_J with $[G_J^{-1} - (-\lambda\phi)]^{-1}$, i.e. $W[J]$ seems to be given by

$$-\frac{1}{2i}\mathrm{Tr}\ln\left(G_J^{-1} + \lambda\phi\right) + \mathcal{K}[(G_J^{-1} + \lambda\phi)^{-1}] = \bar{\mathcal{K}}[\bar{G}]\,. \tag{2.94}$$

Note here that ϕ is the sum of the all distinct connected diagrams with one external point where two propagators meet. [We use here the rules (2.4) and (2.9) in which the propagator $G_J = (\Box + m^2 - J)^{-1}$ is used so that there are only the four-point vertices $-\lambda$ and the pseudovertex does not exist in the graphs of ϕ.] But the above statement is not exactly true, because each element of the graphs of (2.94) is incorrectly weighted. To examine this point the number of the skeletons $N(\bar{\mathcal{K}})$ is defined as follows. Removal of all 1VR vertices in a graph leads to separated graphs which no longer have any lines connecting them. Then all the resulting separated graphs are skeletons and the number of them is $N(\bar{\mathcal{K}})$. Note here that the skeleton and the v_j vertex are slightly different, i.e. v_j does not contain the second and the last term in (2.83) and the trivial skeleton while the skeleton does. An example of the graph of $N(\bar{\mathcal{K}}) = 4$ is given in Fig. 4. Now we see that each graph of $W[J]$ is contained in (2.94) $N(\bar{\mathcal{K}})$ times.

Fig. 4. An example of the graph of $N(\bar{\mathcal{K}}) = 4$.

On the other hand, if we turn our attention to 1VR vertices the graphs of $W[J]$ seem to be generated by

$$\frac{1}{2}\phi(-\lambda)\phi = -\frac{\lambda}{2}\phi^2\,, \tag{2.95}$$

because ϕ is all the distinct connected graphs with one external point [given by the rule (2.9)]. Again this is not true, however, because each element of $W[J]$ appears $N(\mathrm{1VR})$ times, $N(\mathrm{1VR})$ being the number of 1VR vertices in the graph.

Thus the above two ways to construct the graphs of $W[J]$ are not satisfactory. But fortunately we have a simple topological relation:

$$N(\bar{\mathcal{K}}) - N(\mathrm{1VR}) = 1\,. \tag{2.96}$$

This can be proved by noting that the addition of one skeleton having one external point necessarily increases the number of the 1VR vertex by 1. Thus if we take the sum $\bar{\mathcal{K}}[\bar{G}] + \frac{\lambda}{2}\phi^2$, each graph of $W[J]$ is contained exactly once or with correct weight. Hence we have

$$W[J] = \bar{\mathcal{K}}[\bar{G}] + \frac{\lambda}{2}\phi^2 \,. \tag{2.97}$$

This proves (2.79) or Proposition A2″, leading to the proof of Propositions A2 and A2′.

We show below that Proposition A2″ can also be proved by using the summing-up rule, which has been established by the author.[16,1] Indeed we see that Eq. (2.81) is directly obtained by the summing-up rule in the following. If Eq. (2.81) holds, by assuming the form

$$\Gamma = -J[\phi]\phi + \frac{\lambda}{2}\phi^2 + \Delta[\bar{G}] \,, \tag{2.98}$$

we immediately know by differentiation with respect to ϕ that $\Delta[\bar{G}]$ is equal to $\bar{\mathcal{K}}[\bar{G}]$, leading to (2.79).

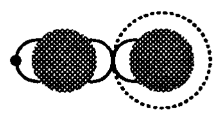
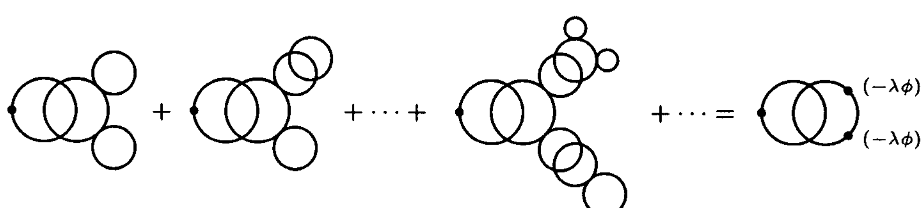


Fig. 5. The 1-part (encircled by the dashed line) in a graph of ϕ.

In order to prove (2.81) we first note that ϕ is all the distinct graphs with one external point [representing the insertion of $\varphi(x)^2$] which are built with the propagators $G_J = (\Box + m^2 - J)^{-1}$ and the four-point vertices $-\lambda$ [the rule (2.9)]. A 1-part is a subdiagram connected to the rest by one four-point vertex. When cut out, the 1-part itself is one element of the graphs of ϕ (see Fig. 5). The summing-up rule is best explained by an example. In short it guarantees that we can sum up the graphs on the left hand side of the following example to the single graph on the right hand side *with correct weight.*



(2.99)

In other words *all the 1-parts directly attached to the skeleton through an external point are summed up to* ϕ. The statement is proved rigorously as follows.

In the graphs of ϕ, we can easily show that *if two different 1-parts have a common part, one completely contains the other.* (Note here that in a vacuum graph this is not true so that the following arguments cannot be applied to the graphs with no external point.) Thus one can *unambiguously* proceed to a larger 1-part starting from one of the 1-parts (which is smaller) in the graph and finally reach *the second-largest 1-part*. See Fig. 6 as an example. (The largest 1-part is the whole graph itself.) This procedure can be repeated to reach the second-largest 1-part starting from another 1-part which is not contained in the former second-largest 1-parts. We continue this until there are no 1-parts other than the second-largest ones. Thereby we find the second-largest 1-part structure of the graph. This operation to find the 1-part structure is done for all the graphs of ϕ. After the operation we sum up all the graphs having the same structure. We thus know that *all the propagators in the graphs are modified to* $\bar{G} = (G_J^{-1} + \lambda\phi)^{-1}$ *while 1VR graphs disappear* because all the second-largest 1-parts are summed up to ϕ with correct weight.

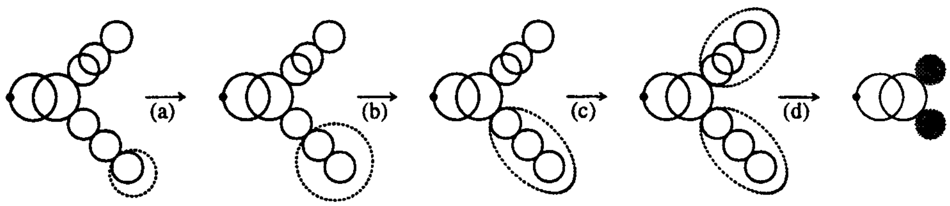


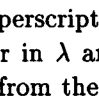
Fig. 6. The procedure to reach the second-largest structure. (a) proceed to a larger 1-part; (b) reach the second-largest 1-part; (c) reach the second-largest 1-parts; (d) reach the second-largest (1-part) structure.

Hence we know that ϕ is all the distinct 1VI graphs (including a derivative of the trivial skeleton) with one external point where propagator $G_J = (\Box + m^2 - J)^{-1}$ is replaced by $\bar{G}$ or

$$\phi = \left.\frac{\delta\bar{\mathcal{K}}[G]}{\delta J}\right|_{J \to J - \lambda\phi}, \tag{2.100}$$

which is equivalent to (2.81). Thus (2.79) or Proposition A2$''$ is justified.

Having shown that Propositions A2, A2$'$ and A2$''$ are true we can take it for granted that Propositions A1 and A1$'$ also hold because Proposition A1$'$ can be regarded as the derivative form of Proposition A2. But Proposition A1 or A1$'$ can be directly proved by using the summing-up rule again. From the rule (2.10) or (2.45) we know that



$$- \quad \Delta J = \left\langle \frac{1}{2}\varphi^2 e^{-\frac{i\lambda}{4!}\varphi^4 + \frac{i}{2}(J^{(1)} + J^{(2)} + \cdots)\varphi^2} \right\rangle_{G^{(0)}}^{\text{excl}}, \tag{2.101}$$

where the superscript excl means that the contributions of the zeroth order and the first order in λ and the derivative of the self-contractions of $J^{(i)}$ with $i \geq 2$ are *excluded* from the expression. The derivative of the self-contractions has been moved on the left hand side. Keeping the graphical meaning of (2.10) in mind we apply the summing-up rule again to obtain

$$- \bullet\!\bigcirc\!\bullet\ \Delta J = \text{all distinct } \textit{1VI graphs with one external point}$$
$$\text{which are built with the propagator}$$
$$\left([G^{(0)}]^{-1} + \lambda\phi\right)^{-1} \text{ and the four-point vertex } -\lambda$$
$$\text{and the pseudovertex } J^{(i)} \text{ with } i \geq 1\,. \qquad (2.102)$$

In the above, all the corrections by the pseudovertex $J^{(1)}\varphi^2$ change the propagator $\left([G^{(0)}]^{-1} + \lambda\phi\right)^{-1}$ back to $G^{(0)}$, and hence we get

$$- \bullet\!\bigcirc\!\bullet\ \Delta J = \left\langle \frac{1}{2}\varphi^2 e^{-\frac{i\lambda}{4!}\varphi^4 + \frac{i}{2}(J^{(2)}+J^{(3)}+\cdots)\varphi^2} \right\rangle_{G^{(0)}}^{\text{1VI/ndself}}$$
$$= \left\langle \frac{1}{2}\varphi^2 e^{-\frac{i\lambda}{4!}\varphi^4 + \frac{i}{2}\Delta J\varphi^2} \right\rangle_{G^{(0)}}^{\text{1VI/ndself}}\,. \qquad (2.103)$$

This equation is, of course, equivalent to Proposition A1′.

The remaining work is to prove Proposition A3′. First the rule (2.91) for ΔJ is easily proved by mathematical induction; we assume the rule is true up to $J^{(n)}$ or the nth order of ΔJ and then we can convince ourselves that the statement for $J^{(n+1)}$ or the $(n+1)$th order of ΔJ is also true from Proposition A1′. For this purpose we have only to note that the graphs of $J^{(n)}$ contain $J^{(i)}$ $(i \leq n-2)$ and have one external point so that the summing-up rule can be applied.

The last task is to prove[b] the rule (2.90) for $\Delta\Gamma$. It is clear from Propositions A2 and A1′ that the graphs appearing in $\Delta\Gamma$ are exhausted in the rule (2.90). Thus it is enough if we confirm that the graphs of $\Delta\Gamma$ in Proposition A2″ appear with the same weight as in the rule (2.90). In other words we justify (2.90) on the basis of Proposition A2″. To this end, we expand $-\frac{1}{2i}\mathrm{Tr}\ln[\bar{G}]^{-1}$ in terms of ΔJ $(= J^{(2)} + J^{(3)} + \cdots)$ and get

$$-\frac{1}{2i}\mathrm{Tr}\ln[\bar{G}]^{-1} - \Delta\mathcal{K}_{\mathrm{tr}} = \sum_{n=2}^{\infty}\frac{1}{2in}\mathrm{Tr}(G^{(0)}\Delta J)^n\,. \qquad (2.104)$$

$\Delta\mathcal{K}_{\mathrm{tr}}$ is canceled by the zeroth and the first order of the expansion. Therefore we get, from the expression $\Delta\Gamma = \mathcal{K}[\bar{G}] - \frac{1}{2i}\mathrm{Tr}\ln[\bar{G}]^{-1} - \Delta\mathcal{K}_{\mathrm{tr}}$,

$$\Delta\Gamma = \mathcal{K}[\bar{G}] + \sum_{n=3}^{\infty}\frac{1}{2in}\mathrm{Tr}\left(G^{(0)}\Delta J\right)^n + \Delta J\ \bullet\!\bigcirc\!\bullet\ \Delta J$$
$$= \mathcal{K}\left[\left([G^{(0)}]^{-1} - \Delta J\right)^{-1}\right] + \sum_{n=3}^{\infty} \bigcirc^{\,n\,\bullet\,\Delta J}_{\,1\,\bullet\,\Delta J}\ (\vdots)$$
$$-\frac{1}{2}\left(-\Delta J\ \bullet\!\bigcirc\!\bullet\ \right) D \left(-\ \bullet\!\bigcirc\!\bullet\ \Delta J\right). \qquad (2.105)$$

[b]The author got the idea of the proof presented below from S. Yokojima, to whom he is very grateful.

By this relation the rule for $\Delta\Gamma$ is also proved by mathematical induction. We assume that the rule is true up to the nth order of $\Delta\Gamma$ or $\Gamma^{(n)}$. We notice here that the first two terms on the right hand side of (2.105) contain each graph $N(v_j)$ times and the last term $N(D)$ times [see the graphical rule (2.91) for ΔJ]. Here $N(v_j)$ and $N(D)$ are the number of v_j vertices ($j = 1, 2, \ldots$) and that of the D propagators respectively. Due to the topological relation

$$N(v_j) - N(D) = 1 \tag{2.106}$$

we confirm that $\Gamma^{(n+1)}$ is given correctly by the final rule (2.90).

3. The Case of the Itinerant Electron Model

In the previous section we have taken the φ^4 theory, which is simple and convenient for developing a general framework. In this section we take a physically more interesting system as another example — the itinerant electron model including the Hubbard model. We couple an external source to the local composite operator corresponding to the spin operator (and to the number density operator). Writing down the effective action for such a system is equivalent to rewriting the theory in terms of the expectation value of the spin operator or the magnetization instead of the external source or the magnetic field. Such a formulation is of course convenient for the study of the magnetic phase of the system — the problem of the spontaneous symmetry breaking of SU(2), which is inherent in the model.

The generating functional for this system (written as Ω in this section instead of W) is a generalization of the thermodynamical potential to the case where an external source, *which depends on imaginary time* τ, is present. This is particularly useful for our purpose and is defined by

$$e^{-\Omega[J]} = \mathrm{Tr}\, T_\tau e^{-\int_0^\beta d\tau \mathcal{H}[J]}\,, \tag{3.1}$$

$$\mathcal{H}[J] = \mathcal{H}_0 + \mathcal{H}_J\,, \tag{3.2}$$

$$\mathcal{H}_0 = \sum_{\mathbf{r}\mathbf{r}'}\sum_{\sigma} t_{\mathbf{r}\mathbf{r}'} a^\dagger_{\mathbf{r}\sigma} a_{\mathbf{r}'\sigma} + U \sum_{\mathbf{r}} n_{\mathbf{r}\uparrow} n_{\mathbf{r}\downarrow}\,, \tag{3.3}$$

$$\mathcal{H}_J = -\sum_{\mathbf{r}\sigma} J_\sigma(\mathbf{r}\tau) n_{\mathbf{r}\sigma} \tag{3.4}$$

$$= -\sum_{\mathbf{r}} h(\mathbf{r}\tau)\hat{S}_z(\mathbf{r}) - \mu N\,, \tag{3.5}$$

where β^{-1} is the temperature of the system and T_τ is the τ-ordering operator. The creation and annihilation operators for the electron of spins σ and σ' at the lattice sites $\mathbf{r}$ and $\mathbf{r}'$ satisfy

$$\{a_{\mathbf{r}\sigma}, a^\dagger_{\mathbf{r}'\sigma'}\} = \delta_{\mathbf{r}\mathbf{r}'}\delta_{\sigma\sigma'}\,. \tag{3.6}$$

Furthermore $t_{\mathbf{r}\mathbf{r}'}$ represents the hopping term and U the on-site Coulomb interaction. We have also introduced

$$n_{\mathbf{r}\sigma} = a^\dagger_{\mathbf{r}\sigma} a_{\mathbf{r}\sigma}\,, \tag{3.7}$$

$$\hat{S}_z(\mathbf{r}) = \frac{1}{2}(n_{\mathbf{r}\uparrow} - n_{\mathbf{r}\downarrow})\,, \tag{3.8}$$

$$N = \sum_{\mathbf{r}} (n_{\mathbf{r}\uparrow} + n_{\mathbf{r}\downarrow})\,, \tag{3.9}$$

$$J_\sigma(\mathbf{r}\tau) = \frac{\sigma}{2} h(\mathbf{r}\tau) + \mu\,. \tag{3.10}$$

We regard below both the chemical potential and the τ-dependent magnetic field $h(\mathbf{r}\tau)$ as external sources for convenience. They are combined to $J_\sigma(\mathbf{r}\tau)$ as in (3.10). Note here that if we want to rewrite the theory in terms of the expectation value of the number density operator without taking the spin operator as another dynamical variable, we have only to set $J_\uparrow = J_\downarrow$ in the following formulae. The spin index σ is defined to take the value $(+1, -1)$ for $(\uparrow, \downarrow)$.

The path integral representation in terms of Grassmann variables z and z^* (corresponding to the operators a and $a^\dagger$ respectively) is given by (see App. D)

$$e^{-\Omega} = \int \mathcal{D}z^* \mathcal{D}z e^{S[z^*, z, J]}\,, \tag{3.11}$$

$$\begin{aligned} S[z^*, z, J] &= -\sum_{xx'\sigma} z^*_{x\sigma} [G_\sigma]^{-1}_{xx'} z_{x'\sigma} - U \sum_x z^*_{x\uparrow} z_{x\uparrow} z^*_{x\downarrow} z_{x\downarrow} \\ &\quad + \sum_{x\sigma} J_{x\sigma} z^*_{x\sigma} z_{x\sigma} \end{aligned} \tag{3.12}$$

$$\equiv -\sum_\sigma z^*_\sigma G^{-1}_{J\sigma} z_\sigma - U z^*_\uparrow z_\uparrow z^*_\downarrow z_\downarrow\,, \tag{3.13}$$

$$\begin{aligned} G^{-1}_{xx'} &= \delta_{\tau\tau'} \left(\delta_{\mathbf{r}\mathbf{r}'} \frac{\partial}{\partial \tau'} + t_{\mathbf{r}\mathbf{r}'} \right)\,, \\ \left[G^{-1}_{J\sigma}\right]_{xx'} &= G^{-1}_{xx'} - \delta_{\tau\tau'} \delta_{\mathbf{r}\mathbf{r}'} J_{x\sigma}\,, \end{aligned} \tag{3.14}$$

where x and x' denote the sets $(\mathbf{r}\tau)$ and $(\mathbf{r}'\tau')$ respectively. From this expression it is straightforward to get the Feynman diagram expansion for Ω in powers of U. The expectation value of the local number operator $n_{\mathbf{r}\sigma}$ is defined as

$$\begin{aligned} \phi_{x\sigma} &= -\frac{\delta \Omega}{\delta J_{x\sigma}} = \langle a^\dagger_{\mathbf{r}\sigma} a_{\mathbf{r}\sigma} \rangle_\tau \\ &= \left\langle \frac{a^\dagger_{\mathbf{r}\uparrow} a_{\mathbf{r}\uparrow} + a^\dagger_{\mathbf{r}\downarrow} a_{\mathbf{r}\downarrow}}{2} + \sigma \frac{a^\dagger_{\mathbf{r}\uparrow} a_{\mathbf{r}\uparrow} - a^\dagger_{\mathbf{r}\downarrow} a_{\mathbf{r}\downarrow}}{2} \right\rangle_\tau = \frac{n_x}{2} - \sigma m_x\,, \end{aligned} \tag{3.15}$$

where x again denotes the set $(\mathbf{r}\tau)$ while n_x and $-m_x$ are the expectation value of the local number operator and the z component of the local spin operator respectively.

The effective action or a generalization of the free energy to the case of τ-dependent dynamical variables is defined by

$$F = \Omega + \int_0^\beta d\tau \sum_{\mathbf{r}\sigma} J_\sigma(\mathbf{r}\tau)\phi_\sigma(\mathbf{r}\tau) \equiv \Omega + \sum_{x\sigma} J_{x\sigma}\phi_{x\sigma}\,, \tag{3.16}$$

with an identity

$$J_{x\sigma} = \frac{\delta F}{\delta \phi_{x\sigma}}\,. \tag{3.17}$$

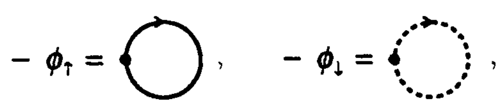
F corresponds to Γ of the previous section. The rule for ϕ corresponding to the rule (2.10) in this case is

$$-\phi_\sigma = \left\langle z_\sigma z_\sigma^* e^{-U z_\uparrow^* z_\uparrow z_\downarrow^* z_\downarrow + \sum_{\sigma'}(J_{\sigma'}^{(1)} + J_{\sigma'}^{(2)} + \cdots) z_{\sigma'}^* z_{\sigma'}} \right\rangle_{G^{(0)}}\,, \tag{3.18}$$

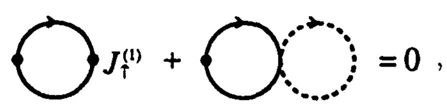
i.e. the sum of all the connected graphs built with four-point vertices U, pseudo-vertices $J_\sigma^{(i)}$ $(i \geq 1)$, and propagators $G_\sigma^{(0)}$ with the notation similar to (2.7). Here $G^{(0)}$ is defined as

$$[G_\sigma^{(0)}]_{xy}^{-1} = G_{xy}^{-1} - \delta_{xy} J_{x\sigma}^{(0)}\,. \tag{3.19}$$

The extra minus sign in (3.18) originates from the sign in $\phi = -\frac{\delta\Omega}{\delta J}$. Then, as mentioned before (below Proposition A1), the inversion formula of the nth order in U is given by the nth order of (3.18) regarding both ϕ_σ and $G_\sigma^{(0)}$ as order unity. Thus we obtain

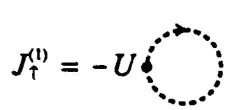


(3.20)



(3.21)

and so on. Here the solid (dashed) line to which an arrow is attached (per loop of lines) represents the propagator of the spin-up (spin-down) electron and it is $G_\uparrow^{(0)}$ $(G_\downarrow^{(0)})$. The dot denotes the place where two propagators meet (corresponding to a derivative with respect to $J^{(0)}$ — note that $\frac{\delta G_\sigma^{(0)}}{\delta J_{\sigma'}^{(0)}} = \delta_{\sigma\sigma'} G_\sigma^{(0)} G_\sigma^{(0)}$). The factor U is associated with a four-point vertex at which spin-up and spin-down propagators come in and out, while no factor is associated with the dot (see App. B). Hence from (3.21) we get

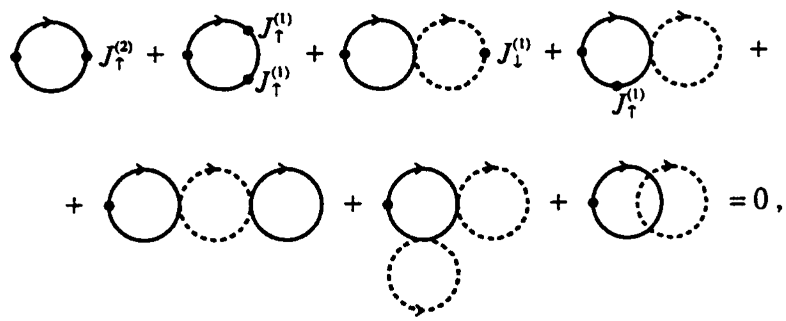


(3.22)

or

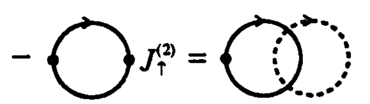
$$J_{-\sigma}^{(1)} = U\phi_\sigma = -U \operatorname{Tr} G_\sigma\,. \tag{3.23}$$

The second order formula of the inversion method is also obtained as that order of (3.18);



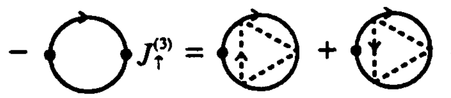
(3.24)

which reduces to, as Eq. (2.37) does to (2.38),



(3.25)

Further, it is easy to find that, corresponding to (2.39),



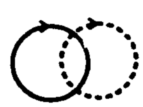
(3.26)

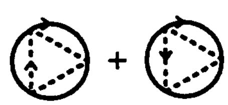
The left hand side of (3.25) or (3.26) can be written as $\frac{\delta\phi_\uparrow}{\delta J_\uparrow^{(0)}} J_\uparrow^{(i)}$, with $i = 2$ or 3. Following the procedure presented in the previous section we get

$$F = F^{(0)} + F^{(1)} + F^{(2)} + F^{(3)} + \cdots , \tag{3.27}$$

$$F^{(0)} = \sum_{x\sigma} J_{x\sigma}^{(0)} \phi_{x\sigma} - \sum_{\sigma} \mathrm{Tr} \ln \left[G_\sigma^{(0)}\right]^{-1} , \tag{3.28}$$

$$F^{(1)} = U \sum_{x} \phi_{x\uparrow}\phi_{x\downarrow} = \frac{U}{2} \sum_{x\sigma} \phi_{x\sigma}\phi_{x-\sigma} , \tag{3.29}$$

$F^{(2)} =$  , (3.30)

$F^{(3)} =$  , (3.31)

where $F^{(n)}$ satisfies

$$J_\sigma^{(n)} = \frac{\delta F^{(n)}}{\delta \phi_\sigma} . \tag{3.32}$$

Note that $J^{(0)}$ contained in $G^{(0)}$ is a functional of ϕ defined by the solution of (3.20) or

$$\phi_{x\sigma} = -G^{(0)}_{xx\sigma} = -\left(\frac{1}{G^{-1} - J^{(0)}_\sigma[\phi]}\right)_{xx} . \tag{3.33}$$

The free energy of the Stoner theory is recreated by $F^{(0)} + F^{(1)}$. Now it is clear that all the propositions given in Sec. 2 hold for the present model with minor and self-evident modifications. Here we repeat them for later convenience.

Proposition B1. The graphical rule for ΔF is given by the equation

$$\Delta F = -\left\langle e^{-Uz^*_\uparrow z_\uparrow z^*_\downarrow z_\downarrow + \sum_\sigma (J^{(2)}_\sigma + J^{(3)}_\sigma + \cdots) z^*_\sigma z_\sigma}\right\rangle^{1\text{VI/nself}}_{G^{(0)}} , \tag{3.34}$$

i.e. the sum of all the connected 1VI/nself diagrams constructed out of four-point vertices, two-point pseudovertices and propagators $G^{(0)}_\sigma$.

Here the 1VI/nself condition implies that only the 1VI graphs are kept and graphs corresponding to the self-contractions of the vertices are excluded.

Proposition B2. $J^{(n)}$ is successively given as a functional of $J^{(0)}_\sigma$ by the formula

$$J^{(n)}_\sigma = n\text{th order of } D_\sigma \times \left\langle z^*_\sigma z_\sigma e^{-Uz^*_\uparrow z_\uparrow z^*_\downarrow z_\downarrow + \sum_{\sigma'} (J^{(2)}_{\sigma'} + J^{(3)}_{\sigma'} + \cdots) z^*_{\sigma'} z_{\sigma'}}\right\rangle^{1\text{VI/ndself}}_{G^{(0)}} , \tag{3.35}$$

where

$$D^{-1}_\sigma = \frac{\delta\phi_\sigma}{\delta J^{(0)}_\sigma} = -\frac{1}{G^{-1} - J^{(0)}_\sigma} \frac{1}{G^{-1} - J^{(0)}_\sigma} . \tag{3.36}$$

Proposition B3. The graphical rule for ΔF is given by the formula

$$\Delta F = -\left\langle e^{-Uz^*_\uparrow z_\uparrow z^*_\downarrow z_\downarrow + \sum_\sigma z^*_\sigma z_\sigma \varphi_{-\sigma}}\right\rangle^{\text{tree/1VI/excl}}_{G^{(0)}, D} , \tag{3.37}$$

or, in a more detailed expression,

$$\begin{aligned} \Delta F &= -\left.\frac{\int \mathcal{D}z^* \mathcal{D}z \mathcal{D}\varphi e^{S_0 + S_{\text{int}}}}{\int \mathcal{D}z^* \mathcal{D}z \mathcal{D}\varphi e^{S_0}}\right|_{\text{conn/tree/1VI/excl}} , \\ S_0 &= -\sum_\sigma z^*_\sigma [G^{(0)}]^{-1} z_\sigma + \frac{1}{2}\sum_\sigma \varphi_\sigma D^{-1}_\sigma \varphi_\sigma , \\ S_{\text{int}} &= -Uz^*_\uparrow z_\uparrow z^*_\downarrow z_\downarrow + \sum_\sigma z^*_\sigma z_\sigma \varphi_{-\sigma} , \end{aligned} \tag{3.38}$$

where the subscript conn/tree/1VI/excl implies that we should take only connected graphs which are tree graphs with respect to the D_σ propagator of the bosonic field

φ_σ and also 1VI with respect to the four-point vertex, and the substructures of the graphs corresponding to $\frac{\delta}{\delta J^{(0)}} \mathrm{Tr} \ln G^{(0)}$ and $\frac{\delta^2}{\delta J^{(0)} \delta J^{(0)}} \mathrm{Tr} \ln G^{(0)}$ are excluded.

Note that Proposition B1 can be deduced from the formula

$$\Delta F = -\left\langle e^{-U z_\uparrow^* z_\uparrow z_\downarrow^* z_\downarrow + \sum_\sigma (J_\sigma^{(2)} + J_\sigma^{(3)} + \cdots) z_\sigma^* z_\sigma} \right\rangle_{G^{(0)}}^{\mathrm{nself}}, \tag{3.39}$$

which is clear from the functional representation:

$$\begin{aligned} e^{-F} &= e^{\sum_\sigma (-J_\sigma^{(0)} \phi_\sigma + \mathrm{Tr} \ln [G_\sigma^{(0)}]^{-1}) - U \phi_\uparrow \phi_\downarrow} \\ &\quad \times \frac{\int \mathcal{D}z^* \mathcal{D}z e^{-\sum_\sigma z_\sigma^* [G_\sigma^{(0)}]^{-1} z_\sigma - U z_\uparrow^* z_\uparrow z_\downarrow^* z_\downarrow + \sum_\sigma [(J_\sigma - J_\sigma^{(0)}) z_\sigma^* z_\sigma - (J_\sigma - J_\sigma^{(0)}) \phi_\sigma] + U \phi_\uparrow \phi_\downarrow}}{\int \mathcal{D}z^* \mathcal{D}z e^{-\sum_\sigma z_\sigma^* [G_\sigma^{(0)}]^{-1} z_\sigma}} \end{aligned} \tag{3.40}$$

or

$$e^{-\Delta F} = \frac{\int \mathcal{D}z^* \mathcal{D}z e^{-\sum_\sigma z_\sigma^* [G_\sigma^{(0)}]^{-1} z_\sigma + (-U z_\uparrow^* z_\uparrow z_\downarrow^* z_\downarrow + U \sum_\sigma \phi_{-\sigma} z_\sigma^* z_\sigma - U \phi_\uparrow \phi_\downarrow) - \sum_\sigma \frac{\delta \Delta F}{\delta \phi_\sigma} (z_\sigma^* z_\sigma - \phi_\sigma)}}{\int \mathcal{D}z^* \mathcal{D}z e^{-\sum_\sigma z_\sigma^* [G^{(0)}]_\sigma^{-1} z_\sigma}}. \tag{3.41}$$

There is another way to state the graph rule. For this purpose $\mathcal{K}[A]$ is defined as follows:

$$\mathcal{K}[A] = \left\langle e^{-U z_\uparrow^* z_\uparrow z_\downarrow^* z_\downarrow} \right\rangle_A^{1\mathrm{VI}}, \tag{3.42}$$

where A is the propagator used in the graphical expression. Then the rule is summarized in the following proposition.

Proposition B1″. $\Delta F = F - (F^{(0)} + F^{(1)})$ is given by $\mathcal{K}[\bar{G}] - \widetilde{\Delta F}$, where

$$\begin{aligned} \bar{G}_\sigma &= \left(G^{-1} - J_\sigma^{(0)} - J_\sigma^{(2)} - J_\sigma^{(3)} - \cdots\right)^{-1} \\ &= \left(G^{-1} - J_\sigma + U \phi_{-\sigma}\right)^{-1}, \end{aligned} \tag{3.43}$$

$$\begin{aligned} \widetilde{\Delta F} &= \sum_\sigma \mathrm{Tr} \ln \left(G^{-1} - J_\sigma^{(0)} - J_\sigma^{(2)} - J_\sigma^{(3)} - \cdots\right) \\ &\quad - \sum_\sigma \mathrm{Tr} \ln \left(G^{-1} - J_\sigma^{(0)}\right) - \sum_\sigma \phi_\sigma \left(J_\sigma - J_\sigma^{(0)} - J_\sigma^{(1)}\right). \end{aligned} \tag{3.44}$$

In other words

$$\begin{aligned} F &= \sum_\sigma \phi_\sigma J_\sigma - U \phi_\uparrow \phi_\downarrow - \sum_\sigma \mathrm{Tr} \ln \bar{G}_\sigma^{-1} + \mathcal{K}[\bar{G}] \\ &\equiv \sum_\sigma \phi_\sigma J_\sigma - U \phi_\uparrow \phi_\downarrow + \bar{\mathcal{K}}[\bar{G}]. \end{aligned} \tag{3.45}$$

4. The Case of QED

The final example is the effective action for the expectation value of gauge-invariant local composite field $\phi^\mu(x) = \langle\bar\psi(x)\gamma^\mu\psi(x)\rangle$ in QED. The practical use of $\Gamma[\phi_\mu]$ in QED is as follows. Although $\langle\bar\psi\gamma^\mu\psi\rangle = 0$ for the vacuum, the lowest relation of the on-shell condition[18] (with the space–time integration over y and the summation over ν suppressed)

$$\Gamma^{(2)}_{\mu x,\nu y}\Delta\phi^\nu(y) = 0\,, \tag{4.1}$$

where

$$\Gamma^{(2)}_{\mu x,\nu y} \equiv \left.\frac{\delta^2\Gamma[\phi]}{\delta\phi_\mu(x)\delta\phi_\nu(y)}\right|_{\phi=0}\,, \tag{4.2}$$

determines the bound state in the channel specified by $\bar\psi\gamma^\mu\psi$. This allows us a *gauge-invariant study of* ${}^3S_1$ of the positronium state. The following work may also be a starting point for the study of the order parameter for the chiral symmetry breaking $\phi = \langle\bar\psi\psi\rangle$ in the massless QED and that of $\langle\bar q^a q^a\rangle$ or $\langle A^a_\mu A^a_\mu\rangle$ in QCD. Here q and A^a_μ are operators for quarks and gluons respectively. All these are believed to be nonvanishing objects, in contrast to $\bar\psi\gamma^\mu\psi$. The lowest order discussion of $\langle\bar\psi\psi\rangle$ has been given in Ref. 11.

The generating functional in this case is given by (with the space–time integration and the summation over the Greek index suppressed)

$$e^{iW[J,K]} = \int \mathcal{D}\bar\psi\mathcal{D}\psi\mathcal{D}A e^{iS[\bar\psi\psi AJ]}\,, \tag{4.3}$$

$$\begin{aligned} S[\bar\psi,\psi,A,J] &= -\bar\psi G^{-1}\psi - \frac{1}{2}A^\mu D^{-1}_{\mu\nu}A^\nu + e\bar\psi\gamma_\mu\psi A^\mu + J_\mu\bar\psi\gamma^\mu\psi \\ &= -\bar\psi G_J^{-1}\psi - \frac{1}{2}A^\mu D^{-1}_{\mu\nu}A^\nu + ej_\mu A^\mu\,, \end{aligned} \tag{4.4}$$

where

$$G^{-1} = -i\gamma_\mu\partial^\mu + m\,, \tag{4.5}$$

$$G_J^{-1} = G^{-1} - J_\mu\gamma^\mu\,, \tag{4.6}$$

$$D^{-1}_{\mu\nu} = -\Box g_{\mu\nu} + \left(1 - \frac{1}{\lambda}\right)\partial_\mu\partial_\nu\,, \tag{4.7}$$

$$j_\mu = \bar\psi\gamma_\mu\psi\,. \tag{4.8}$$

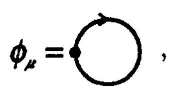
Here the parameter λ specifies the gauge. Then we get Feynman graphs for $\phi_\mu = \frac{\delta W}{\delta J^\mu} = \langle j_\mu\rangle = \langle\bar\psi\gamma_\mu\psi\rangle$:

$$\phi_\mu = \langle\bar\psi\gamma_\mu\psi e^{e\bar\psi\gamma_\mu\psi A^\mu + (J^{(1)}_\mu + J^{(2)}_\mu + \cdots)\bar\psi\gamma^\mu\psi}\rangle_{G^{(0)},D}\,, \tag{4.9}$$

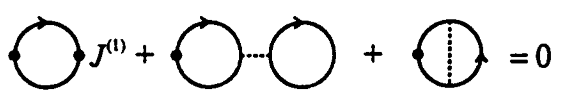
i.e. the sum of all the connected graphs built with three-point vertices ($\bar{\psi}\gamma^\mu\psi A_\mu$), two-point pseudovertices ($J_\mu^{(i)}\bar{\psi}\gamma^\mu\psi$), electron propagators $G^{(0)}$ and photon propagators D. Here $J_\mu^{(i)}$ is the ith order (in e^2) of J^μ and the propagator $G^{(0)}$ is G_J but J replaced by $J^{(0)}$. The quantity $J^{(0)}$ is defined by

$$\phi_\mu(x) = i\gamma_\mu G^{(0)}_{\mu\mu}(x,x)\,, \tag{4.10}$$

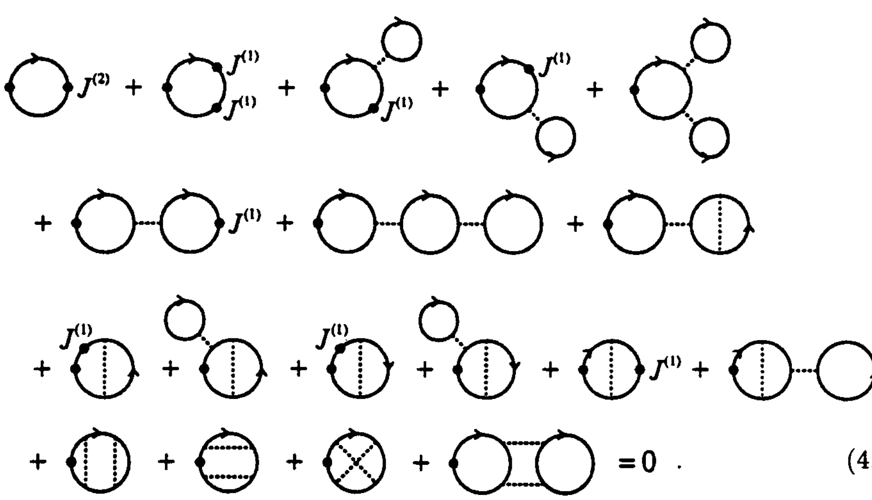
which is equivalent to (4.11). By writing down the ith order of (4.9) one gets the inversion formula of that order. For example,



(4.11)



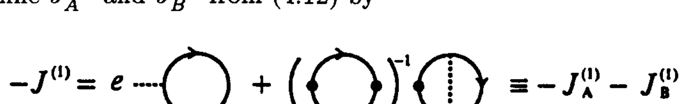
(4.12)



(4.13)

In the above graphs we associate the electron propagator $G^{(0)}$ with each solid line and the photon propagator D with each dashed line. In addition a factor $e\gamma_\mu$ or γ^μ is assigned to a vertex and a dot (•) respectively (see App. B).

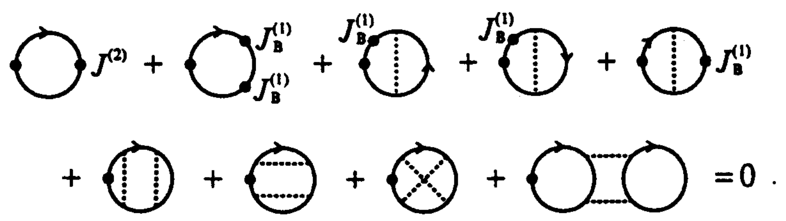
If we define $J_A^{(1)}$ and $J_B^{(1)}$ from (4.12) by



(4.14)

we see that all the $J_A^{(1)}$'s exactly cancel out the 1PR structure appearing in the ith order of (4.9) with $i \geq 2$. Here 1PR means one-particle-reducible *with respect to the photon propagator*. Hereafter the 1PI graph is defined as the graph which is not

1PR *in the photon channel.* Indeed all the 1PR graphs in (4.13) disappear after substitution of the last equation due to $J_A^{(1)}$ while $J_B^{(1)}$ remains;

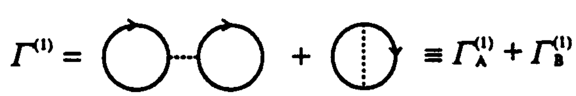


(4.15)

The effective action in this case is defined by $\Gamma = W - J_\mu \phi^\mu$ (with the space–time integration suppressed) as usual with an identity $-J^\mu = \frac{\delta \Gamma}{\delta \phi_\mu}$. Thus integrating (4.15) one can obtain $\Gamma^{(2)}$ [and higher order of Γ by using (4.9)]. Here we can take another course instead. For this purpose let us first examine the path integral representation of Γ. Integrating out the photon field we get

$$e^{i\Gamma} = \int \mathcal{D}\bar{\psi}\mathcal{D}\psi e^{-i\bar{\psi}G^{-1}\psi + i\frac{e^2}{2}j_\mu D^{\mu\nu} j_\nu + iJ_\mu j^\mu - iJ_\mu \phi^\mu} . \tag{4.16}$$

Since $\Gamma^{(i)}$ is defined by $-J_\mu^{(i)} = \frac{\delta \Gamma^{(i)}}{\delta \phi^\mu}$ the quantities $\Gamma_A^{(1)}$ and $\Gamma_B^{(1)}$ are defined as



(4.17)

in accordance with (4.14). The quantities ΔJ and $\Delta\Gamma$ in this case are expanded as

$$\Delta J = J_B^{(1)} + J^{(2)} + J^{(3)} + \cdots , \tag{4.18}$$

$$\Delta\Gamma = \Gamma_B^{(1)} + \Gamma^{(2)} + \Gamma^{(3)} + \cdots . \tag{4.19}$$

Noting that $\Gamma_A^{(1)} = \frac{e^2}{2}\phi^\mu D_{\mu\nu}\phi^\nu$ and $\Gamma^{(0)} = -J_\mu^{(0)}\phi^\mu - i \,\mathrm{Tr}\ln[G^{(0)}]^{-1}$, we get

$$\begin{aligned} e^{i\Gamma} &= e^{i\left(-J_\mu^{(0)}\phi^\mu - i\,\mathrm{Tr}\ln\left[G^{(0)}\right]^{-1}\right) + i\frac{e^2}{2}\phi^\mu D_{\mu\nu}\phi^\nu} \\ &= \frac{\int \mathcal{D}\bar{\psi}\mathcal{D}\psi e^{-i\bar{\psi}\left[G^{(0)}\right]^{-1}\psi + i\frac{e^2}{2}j^\mu D_{\mu\nu}j^\nu + i(J_\mu - J_\mu^{(0)})j^\mu - i(J_\mu - J_\mu^{(0)} + \frac{e^2}{2}\phi^\nu D_{\nu\mu})\phi^\mu}}{\int \mathcal{D}\bar{\psi}\mathcal{D}\psi e^{-i\bar{\psi}\left[G^{(0)}\right]^{-1}\psi}} \end{aligned} \tag{4.20}$$

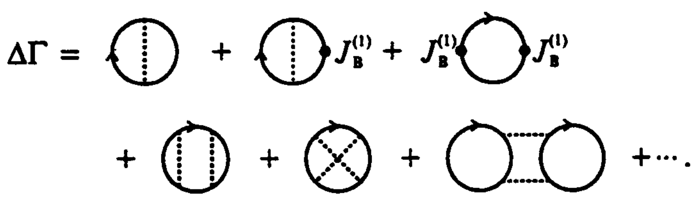
or

$$e^{i\Delta\Gamma} = \frac{\int \mathcal{D}\bar{\psi}\mathcal{D}\psi e^{i\left[-\bar{\psi}\left[G^{(0)}\right]^{-1}\psi + e^2\left(\frac{1}{2}j^\mu D_{\mu\nu}j^\nu - \phi^\mu D_{\mu\nu}j^\mu\right) + \frac{1}{2}\phi^\mu D_{\mu\nu}\phi^\nu - \frac{\delta\Delta\Gamma}{\delta\phi^\mu}(j^\mu - \phi^\mu)\right]}}{\int \mathcal{D}\bar{\psi}\mathcal{D}\psi e^{-i\bar{\psi}\left[G^{(0)}\right]^{-1}\psi}} . \tag{4.21}$$

We write (4.21) as

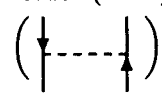
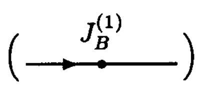
$$\Delta\Gamma[\phi] = \frac{1}{i}\left\langle e^{i\frac{e^2}{2}j_\mu D^{\mu\nu}j_\nu - i\frac{\delta\Delta\Gamma}{\delta\phi^\mu}j_\mu}\right\rangle_{G^{(0)}}^{\mathrm{nself}} . \tag{4.22}$$

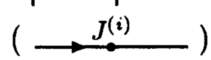
The meaning of nself is that we have to exclude the self-contraction of the electron propagators. By using (4.22) and noting the cancellation similar to that in (4.13) we get



(4.23)

Combined with the arguments similar to those of previous subsections, we arrive at the following proposition (with a similar statement for a graph of $J^{(n)}$):

Proposition C. $\Gamma^{(n)}$ $(n \geq 2)$ is the sum of all possible nth order (in e^2) 1PI diagrams constructed out of the four-point vertex of order e^2 , the vertex of order e^2 

and vertices of order e^{2i} 

$(2 \leq i < n)$. Here the propagator is $G^{(0)}$. In other words

$$\Delta\Gamma[\phi] = \frac{1}{i}\left\langle e^{\frac{ie^2}{2} j_\mu D^{\mu\nu} j_\nu + i(J^{(1)}_{B\mu}+J^{(2)}_\mu+J^{(3)}_\mu+\cdots)j^\mu}\right\rangle^{1\mathrm{PI}}_{G^{(0)}}, \tag{4.24}$$

where $\langle\cdots\rangle_{1\mathrm{PI}}$ means 1PI (in terms of the photon lines) connected Wick contraction using the propagators $G^{(0)}$ which is a functional of $J^{(0)}$.

Of course, there are various equivalent modifications of Proposition C.

5. Discussion — The Renormalization Problem

The problem of the renormalization of composite operators has been studied by many authors.[19] They considered the renormalization of the expectation value of the operators which are the product of local composite operators and elementary field operators such as $\varphi(x_1)^2 \cdots \varphi(x_l)^2\varphi(y_1)\cdots\varphi(y_n)$. But the main concern of the usual approach is somewhat different from what is required in this paper and the usual scheme is not readily applicable to our case. So we suggest here a renormalization scheme which is appropriate to our problem. To make the point clear, we show an example of the renormalization of the composite operator in the context of the inversion method for a simple model. For this purpose, we consider the two-dimensional Gross–Neveu model[8] with the source term

$$\mathcal{L} = \bar\psi(x)i\partial\!\!\!/\psi(x) + \frac{1}{2}g^2\left(\bar\psi(x)\psi(x)\right)^2 + J\bar\psi(x)\psi(x), \tag{5.1}$$

where ψ is the N-component, massless fermion field. We show that the effective action of the local composite operator $\bar\psi(x)\psi(x)$ can be made finite at the level of the first order (in g^2) of the inversion series. This is accomplished by redefining the normalization of the composite operator $\bar{\psi}(x)\psi(x)$. As seen below, the result is consistent with that of the original paper,[8] where the effective action of the auxiliary field σ is renormalized.

We assume that the renormalized composite operator $\left(\bar{\psi}(x)\psi(x)\right)_R$ is related to the bare composite operator $\bar{\psi}(x)\psi(x)$ by

$$\bar{\psi}\psi = Z(\bar{\psi}\psi)_R\,, \tag{5.2}$$

where the Z factor is expanded in terms of the coupling constant g^2:

$$Z = 1 + g^2 Z^{(1)} + \cdots \tag{5.3}$$

Introducing the generating functional in the usual way, we define the variable ϕ in this case as follows:

$$Z\phi = \frac{dW}{dJ} = Z\langle(\bar{\psi}\psi)_R\rangle^J = \phi^{(0)}(J) + \phi^{(1)}(J) + \cdots . \tag{5.4}$$

The essential point is that the inversion is performed between J and the above rescaled ϕ. In the large N limit $\phi^{(0)}$ and $\phi^{(1)}$ are given through Feynman graph:

$$\phi^{(0)}(J) = \frac{N}{i}\int \frac{d^2p}{(2\pi)^2}\mathrm{tr}\frac{1}{\not p - J} = \frac{NJ}{2\pi}\ln\frac{\Lambda^2 + J^2}{J^2}\,, \tag{5.5}$$

$$\begin{aligned}\phi^{(1)}(J) &= g^2N^2\int \frac{d^2p}{(2\pi)^2}\mathrm{tr}\frac{1}{\not p - J}\int \frac{d^2k}{(2\pi)^2}\mathrm{tr}\frac{1}{(\not p - J)^2} \\ &= -g^2\phi^{(0)}(J)\phi^{(0)\prime}(J)\,.\end{aligned} \tag{5.6}$$

Here we have introduced the cutoff momentum Λ. Now we apply the inversion method, by considering ϕ (not $Z\phi$) as order unity, to obtain

$$\phi = \phi^{(0)}\left(J^{(0)}(\phi)\right), \tag{5.7}$$

$$g^2Z^{(1)}\phi = \phi^{(0)\prime}\left(J^{(0)}(\phi)\right)J^{(1)}(\phi) + \phi^{(1)}\left(J^{(0)}(\phi)\right), \tag{5.8}$$

from which we get

$$J^{(1)}(\phi) = \frac{dJ^{(0)}(\phi)}{d\phi}g^2Z^{(1)}\phi + g^2\phi\,. \tag{5.9}$$

Here we have used Eq. (5.6) and the fact that $\left[\phi^{(0)\prime}(J^{(0)}(\phi))\right]^{-1} = dJ^{(0)}(\phi)/d\phi$. Then the inversion series can be written as

$$J = J^{(0)}(\phi) + J^{(1)}(\phi) + \cdots = J^{(0)}\left(\phi + g^2Z^{(1)}\phi\right) + g^2\phi + O[g^4]\,. \tag{5.10}$$

Noting that $J^{(0)}$ is the inverse of $\phi^{(0)}$ [see (5.7)] we arrive at (up to the order of g^2)

$$\phi + g^2Z^{(1)}\phi = \phi^{(0)}(J - g^2\phi) = -\frac{N}{2\pi}(J - g^2\phi)\ln\frac{\Lambda^2 + (J - g^2\phi)^2}{(J - g^2\phi)^2}\,. \tag{5.11}$$

If we define

$$Z^{(1)} = \frac{N}{2\pi}\left[\ln\frac{\Lambda^2}{(J-g^2M)^2} - 2\right], \tag{5.12}$$

ϕ becomes a finite quantity and for $J=0$ we obtain

$$\phi = \frac{Ng^2\phi}{\pi}\left(\ln\frac{M}{\phi} + 1 + O\left[\frac{1}{\Lambda^2}\right]\right), \tag{5.13}$$

or, regarding Λ as infinity,

$$\phi = Me^{1-\pi/\lambda}, \tag{5.14}$$

with $\lambda = Ng^2$. This expression agrees with (4.18) of Ref. 8. Note here that if we set $J=0$ the expectation value of ϕ and that of the auxiliary field σ coincide with each other. This suggests that our prescription of renormalization in the inversion scheme is a correct one. Now the effective action up to the first order in g^2 becomes

$$\Gamma = \frac{1}{Zi}\mathrm{Tr}\left[(i\partial\!\!\!/ + J^{(0)}(Z\phi)\right] - \phi J^{(0)}(Z\phi) - \frac{g^2}{2}\phi^2 + O[g^2], \tag{5.15}$$

where Z is given by (5.3) and (5.12). In spite of its appearance the above Γ is finite up to g^2 of course. Whether we can remove the divergences by the Z factor in (5.2) at higher order is now under study.

Acknowledgments

I would like to thank Prof R. Fukuda and Dr S. Yokojima for suggestions, discussions and encouragement throughout the work.

Appendix A. Legendre Transformation and the Inversion Method

In this appendix we look more carefully at the reason why we should assume that ϕ is of order $\lambda^0 = 1$ or independent of λ in our inversion process. This point has been exemplified in terms of diagrams. The explanation was not necessarily familiar to everyone. Here we present a clear explanation in purely mathematical language. Although the following discussion is trivial it is worthwhile in order to understand the foundation of the inversion method. For brevity the case of x-independent variables J and ϕ is considered.

Consider the quantity $W[J,\lambda] - J\phi[J,\lambda]$, in which $\phi[J,\lambda] \equiv \frac{\delta W[J,\lambda]}{\delta J}$. Here we have emphasized the λ dependence. If we take a small variation of this quantity *assuming that J and λ are independent variables*, it becomes

$$\begin{aligned}\frac{\delta W[J,\lambda]}{\delta J}dJ &+ \frac{\delta W[J,\lambda]}{\delta\lambda}d\lambda - dJ\phi[J,\lambda] - Jd\phi[J,\lambda]\\ &= \frac{\delta W[J,\lambda]}{\delta\lambda}d\lambda - Jd\phi[J,\lambda]\,.\end{aligned} \tag{A.1}$$

Hence we see that the quantity can be regarded as a function(al) of two independent variables ϕ and λ. We thus write the quantity $W - J\phi$ as $\Gamma[\phi, \lambda]$. What is implied here is as follows: if we solve the relation $\phi = \frac{\delta W[J,\lambda]}{\delta J}$ in favor of J *assuming that the two quantities* ϕ *and* λ *are mutually independent* to get $J = J[\phi, \lambda]$ and then insert this expression of J into all J appearing in $W[J] - J\phi$, then $W[J] - J\phi$ is automatically written by only two independent variables ϕ and λ. In other words, the inversion process of Legendre transformation is carried out regarding ϕ as independent of λ. Hence the process in the inversion method exactly coincides with the inversion process of Legendre transformation. Note that once the inversion or Legendre transformation is performed and after the sources are set to the desired values, zero for example, the resultant ϕ depends on λ of course.

Appendix B. Feynman Rules

φ^4 *theory*

Although well known, we summarize for clarity the rule (Rule A) to get algebraic expressions from the corresponding graphs for the φ^4 theory.

Rule A1. In one specific way (as one likes), assign n labels $x_1, \dots, x_n$ (internal points) to all the four-point vertices and the pseudovertices, where n is the total number of vertices (including the pseudovertex).

Rule A2. Associate a propagator G_J [for the rules (2.4) and (2.9)] or $G^{(0)}$ [for the rules (2.5) and (2.10)] with each line. A factor $-\lambda$ and $J^{(i)}$ are assigned to the four-point vertex and the pseudovertex of the ith order respectively. No factor is assigned to the dot which corresponds to the external point.

Rule A3. Associate a factor i^{-L} for a diagram, where L is the number of independent momenta of the graph.

Rule A4. Associate a symmetry factor S for a diagram.

Rule A5. Sum (integrate) the product of all factors in Rules A2–A4 over the space–time index $x_1, \dots, x_n$.

The symmetry factor S for each graph is given by the line symmetry number S_L and the vertex symmetry number S_V as $S = \frac{1}{S_L} \cdot \frac{1}{S_V}$. As is well known, S_L and S_V are obtained through the following rule:

Rule $\mathbf{S_L1}$. If there is a line which starts from a vertex (including the dot • and pseudovertex) and comes back directly to the starting vertex, associate the factor 2.

Rule $\mathbf{S_L2}$. If there are m lines ($m = 2, 3, 4$) directly connecting two common vertices (including the pseudovertex), associate the factor $m!$.

Rule $\mathbf{S_L3}$. The product of all the factors in Rules $\mathrm{S_L}1$ and $\mathrm{S_L}2$ is S_L.

Rule $\mathbf{S_V}$. Assign n labels $1, \dots, n$ to n vertices (including the pseudovertex) in an arbitrary way. Count the number of all possible other ways of assigning n labels that give the same topological structure as the first specific way. The number thus obtained plus 1 is S_V.

For definiteness we give some examples; the graph which appeared in (2.19) has (S_L, S_V)=$(2,1)$: three graphs of (2.21) have $(2!^2{\cdot}2, 1)$, $(2^2, 2)$ and $(3!, 2)$ respectively.

As another example we consider the reduction of (2.37) to (2.38). Since the symmetry factors of the second, fourth and sixth graphs on the left hand side of (2.37) are $(S_L, S_V) = (1,2)$, $(2,1)$ and $(2^2, 2)$, the contribution of the three graphs becomes zero. This is because, after we replace $J^{(1)}$ by the use of (2.35) (whose symmetry factor is 2), new symmetry factors of these graphs become $1 \cdot 2 \cdot 2^2$ $2 \cdot 1 \cdot 2$ and $2^2 \cdot 2$ respectively. By a similar argument we find the cancellation of the third and fifth graphs on the left hand side of (2.37). Thus we get (2.38) from (2.37).

The itinerant electron model

The rules for the itinerant electron model are given as follows (Rule B). Rules B1, B4 and B5 are the same as Rules A1, A4 and A5 respectively. Rule B3 is Rule A3 with i^{-L} replaced by $(-1)^L(-1)^{L_f}$, where L_f is the number of fermion loops. Rule A2 is changed into:

Rule B2. Associate y——▶——x and y- - ▶ - -x with $\left[G_\uparrow^{(0)}\right]_{xy}$ and $\left[G_\downarrow^{(0)}\right]_{xy}$ respectively, and the factor U is assigned to the four-point vertex. The factor $J_\sigma^{(i)}$ is also associated with the pseudovertex of the ith order. No factor is assigned to the external point.

As for the symmetry factor $S\left(= \frac{1}{S_L} \cdot \frac{1}{S_V}\right)$, rules for S_L and S_V are essentially the same as those of the φ^4 theory except for the fact that we have to distinguish the spin-up and spin-down propagators and their directions of the arrows when we consider the topological equivalence. Thus the factor S_L is always 1 in this model.

QED

Finally, the rules for QED are presented as follows:

Rule C1. Assign n labels, in one specific way as one likes, $(x_1, \mu_1), \ldots, (x_n, \mu_n)$, to vertices (including pseudovertices).

Rule C2. Associate an electron propagator $G^{(0)}$ with each solid line and a photon propagator D with each dashed line.

Rule C3. Associate a factor $e\gamma_\mu$ and $J_\mu^{(i)}\gamma^\mu$ with the three-point vertex and the pseudovertex respectively. γ^μ is assigned to the dot ($\bullet$) which corresponds to the external point.

Rule C4. Associate a factor $i^{-L}(-1)^{L_f}$, where L is the number of loop momenta of the graph and L_f is the number of the fermion loops.

Rule C5. Associate a symmetry factor S for a diagram.

Rule C6. Sum the product of all the factors in Rules C2–C5 over $x_1 \cdots x_n$ and $\mu_1 \cdots \mu_n$.

The symmetry factors are calculated as before. Note that S_L is always 1 in QED.

Appendix C. The Inversion Method for $\langle\varphi(x)\rangle$ and $\langle\varphi(x)\varphi(y)\rangle$

We show below how the inversion method works to reproduce well-known results for the effective action of $\langle\varphi(x)\rangle$ and $\langle\varphi(x)\varphi(y)\rangle$. For simplicity we consider the φ^4 theory, and several lower orders of the known rule are explicitly studied rather than giving formal proof.

The case of $\langle\varphi(x)\rangle$

In order to study the effective action of elementary field $\phi(x)$, the generating functional $W[J]$ is defined as in (2.1) with $S[\varphi, J]$ replaced by

$$S[\varphi, J] = -\frac{1}{2}\int d^4x\varphi(x)(\Box + m^2)\varphi(x) - \frac{\lambda}{4!}\int d^4x\varphi(x)^4 + \int d^4xJ(x)\varphi(x)\,. \tag{C.1}$$

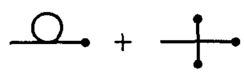
The dynamical variable ϕ for the effective action is

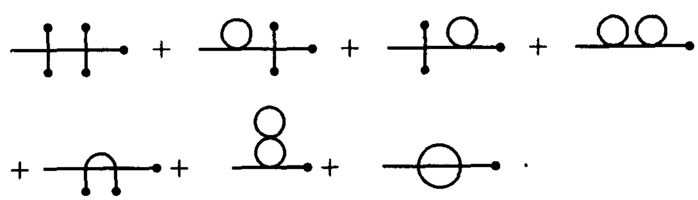
$$\phi(x) = \frac{\delta W}{\delta J(x)} \equiv \langle\varphi(x)\rangle^J\,, \tag{C.2}$$

by the use of which $\Gamma[\phi]$ is defined by (2.11) and Eq. (2.12) holds as an identity.

Now the original series expansion in λ is given by (suppressing the x dependence)

$\phi^{(0)} =$, (C.3)

$\phi^{(1)} =$  , (C.4)

$\phi^{(2)} =$  . (C.5)

Here a dot denotes the external source J and a line the propagator $\frac{1}{\Box+m^2}$. Thus (2.25), the right hand side of which is (C.3) with J replaced by $J^{(0)}$, becomes

$\phi =$ (C.6)

or

$$\phi(x) = \left(\frac{1}{\Box + m^2}\right)_{xy} J^{(0)}(y)\,, \tag{C.7}$$

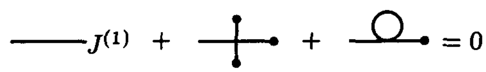
from which $J^{(0)}$ is obtained explicitly as opposed to the case of the local composite operators;

$$J^{(0)} = (\Box + m^2)_{xy}\phi(y)\,. \tag{C.8}$$

Hereafter, all the dots in the graphs denote $J^{(0)}$ instead of J as in (C.6). We immediately know that

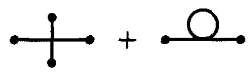
$$\Gamma^{(0)} = -\frac{1}{2}\phi(\Box + m^2)\phi \tag{C.9}$$

by integrating $J^{(0)} = -\frac{\delta\Gamma^{(0)}}{\delta\phi}$. From (C.3) and (C.4), the inversion formula of order λ, (2.26), becomes

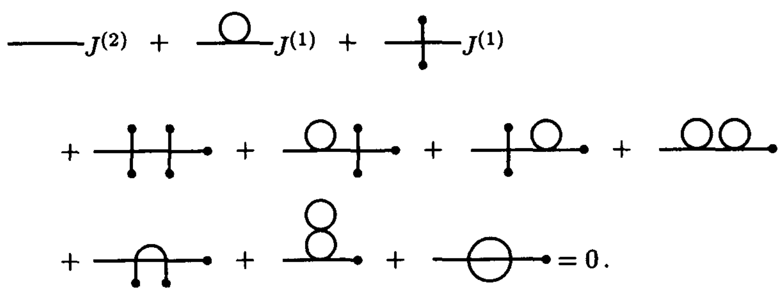


(C.10)

if we note that $\phi^{(0)\prime} = \left(\Box + m^2\right)^{-1}$, which is denoted by a line. The integration of $J^{(1)} = -\frac{\delta\Gamma^{(1)}}{\delta\phi}$ leads to

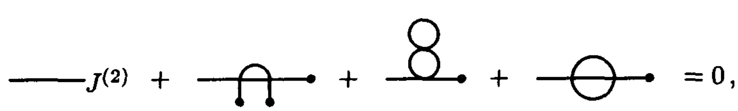
$\Gamma^{(1)} =$  . (C.11)

By (C.6) we confirm that $\Gamma^{(1)}$ is a functional of ϕ indeed. Equation (C.9) and the first term of (C.11) constitute the usual tree part of the one-particle-irreducible (1PI) effective action. From (C.3) to (C.5), the second order formula (2.27) is written as



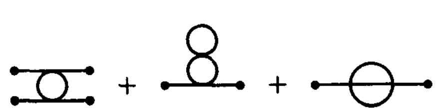
(C.12)

The second term of (2.27) disappears because $\phi^{(0)\prime\prime}[J^{(0)}] = 0$. Using (C.10) we see that the one-particle-reducible (1PR) graphs in (C.12) exactly cancel out each other to yield



(C.13)

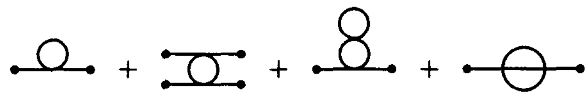
from which we obtain

$\Gamma^{(2)} =$  . (C.14)

This course of study can be continued up to the desired order to give the well-known result

$$\Gamma = -\frac{1}{2}\phi(\Box + m^2)\phi - \frac{\lambda}{4!}\phi^4 + \mathcal{K}_{1\mathrm{PI}}[\phi]\,, \tag{C.15}$$

where $\mathcal{K}_{1\mathrm{PI}}[\phi]$ is the 1PI vacuum graph $\mathcal{K}_{1\mathrm{PI}}[(\Box+m^2)^{-1}J]$ (written in terms of the original J representation) but with $(\Box+m^2)^{-1}J$ replaced by ϕ or

$\mathcal{K}_{1\mathrm{PI}}[\phi] =$  $+\cdots$ (C.16)

with the notation (C.6). We note here that without using (C.3)–(C.5) we can directly obtain (C.10), (C.12) and higher order relations if we note the equation corresponding to (2.10). This point is taken in the following case of $\langle\varphi(x)\varphi(y)\rangle$. It is easy to convince oneself that if one uses $(\Box+m^2+\lambda\phi^2/2)^{-1}$ instead of $(\Box+m^2)^{-1}$ then the result of Ref. 5 is obtained.

The case of $\langle\varphi(x)\varphi(y)\rangle$

Now we consider the effective action of the bilocal composite operator. The generating functional $W[J]$ in this case is defined as in (2.1) with $S[\varphi, J]$ replaced by

$$\begin{aligned} S[\varphi, J] &= -\frac{1}{2}\int d^4x\varphi(x)(\Box+m^2)\varphi(x) - \frac{\lambda}{4!}\int d^4x\varphi(x)^4 \\ &\quad + \frac{1}{2}\int d^4xd^4yJ(x,y)\varphi(x)\varphi(y) \\ &\equiv -\frac{1}{2}\varphi G_J^{-1}\varphi - \frac{\lambda}{4!}\varphi^4\,, \end{aligned} \tag{C.17}$$

$$G_J^{-1} \equiv G_J^{-1}(x,y) = (\Box+m^2)\delta^4(x-y) - J(x,y)\,. \tag{C.18}$$

Note here that $J(x,y)$ has been absorbed in the propagator G_J. We define $\phi(x,y)$ and $\Gamma[\phi]$ by

$$\phi(x,y) = \frac{\delta W}{\delta J(x,y)} \equiv \frac{1}{2}\langle\varphi(x)\varphi(y)\rangle\,, \tag{C.19}$$

$$\Gamma[\phi] = W[J] - \int d^4xd^4yJ(x,y)\phi(x,y)\,. \tag{C.20}$$

Then the zeroth order inversion formula (2.25), which is directly obtained as the zeroth order of the relation corresponding to (2.10), gives (with the space–time index omitted)

$$\phi = \text{———}\quad . \tag{C.21}$$

The line denotes the propagator G evaluated at $J = J^{(0)}$, namely

$$\phi = \frac{1}{i}\frac{1}{\Box+m^2-J^{(0)}} \equiv \frac{1}{i}G^{(0)}\,. \tag{C.22}$$

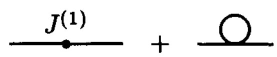
Unlike the local case the key point is that this relation can be explicitly inverted to give $J^{(0)}$ [cf. (2.16)], i.e.

$$J^{(0)} = \Box + m^2 + i\phi^{-1}\,, \tag{C.23}$$

which gives, by integration,

$$\Gamma^{(0)} = \mathrm{Tr}(\Box + m^2)\phi + i\ \mathrm{Tr}\ln\phi\,. \tag{C.24}$$

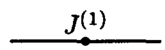
Equation (2.26) or the inversion formula of order λ, which is obtained by the first order of the equation like (2.10), gives

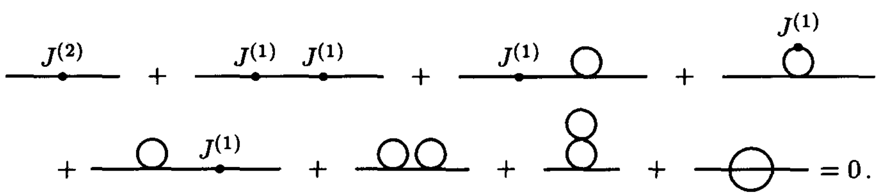


(C.25)

or, through integration,

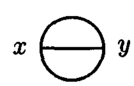
$\Gamma^{(1)} =$  . (C.26)

We have used the notation in which  stands for $G^{(0)}_{xz} J^{(1)}_{zw} G^{(0)}_{wy}$ where $\frac{1}{i}G^{(0)} = \phi$ [see (C.22)]. From (C.26) we make sure that $\Gamma^{(1)}$ is actually a functional of the *bilocal* variable ϕ because lines in the graphs represent ϕ. The second order formula (2.27) given by the equation like (2.10) is written as



(C.27)

Using (C.25) we see that the one- or two-particle-reducible (2PR) graphs in (C.27) exactly cancel out to give

$J^{(2)}(x,y) =$ 

(C.28)

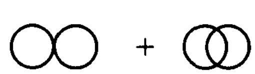
or

$\Gamma^{(2)}(x,y) =$  . (C.29)

As in the case of $\langle\varphi(x)\rangle$, we can continue the process and get the well-known result

$$\Gamma = \mathrm{Tr}\left(\Box + m^2\right)\phi + i\ \mathrm{Tr}\ln\phi + \mathcal{K}_{\mathrm{2PI}}[\phi]\,, \tag{C.30}$$

where $\mathcal{K}_{\mathrm{2PI}}[\phi]$ is the original 2PI graph $\mathcal{K}_{\mathrm{2PI}}\left[\frac{1}{i}\frac{1}{\Box+m^2}\right]$ with $\frac{1}{i}\frac{1}{\Box+m^2}$ replaced by ϕ or

$\mathcal{K}_{\mathrm{2PI}}[\phi] =$  $+\cdots\,.$ (C.31)

Appendix D. Path Integral Formula for the Fermion Coherent State

In this appendix we derive (3.11) from (3.1). In order to clarify the notations, we first enumerate some formulae for the fermionic coherent state in the case of a single mode. The generalization to the multimode case is straightforward. For the anticommuting operator a, $a^\dagger$ like (3.6), the coherent state is defined as

$$a|z\rangle = z|z\rangle\,, \qquad \langle z|a^\dagger = \langle z|z^*\,, \tag{D.1}$$

where z and z^* are Grassmann numbers. Then inner product of the two states becomes

$$\langle z|z'\rangle = e^{z^* z'}\,, \tag{D.2}$$

which means that the coherent state is neither normalized nor orthogonalized. The matrix element in the coherent state is

$$\langle z|\mathcal{O}(a^\dagger, a)|z'\rangle = \mathcal{O}(z^*, z')e^{z^* z'}\,, \tag{D.3}$$

where $\mathcal{O}$ is a *normal-ordered* operator. The overcompleteness is expressed as

$$\int dz^* dz e^{-z^* z}|z\rangle\langle z| = 1\,. \tag{D.4}$$

The trace of a normal-ordered operator becomes

$$\mathrm{Tr}\ \mathcal{O}(a^\dagger, a) = \int dz^* dz e^{-z^* z}\langle -z|\mathcal{O}(a^\dagger, a)|z\rangle\,. \tag{D.5}$$

In order to derive (3.11), we first estimate

$$\langle z_F|T_\tau e^{-\int d\tau (t_{\alpha\beta}a^\dagger_\alpha a_\beta + \mathcal{V}(a^\dagger_\gamma a_\gamma))}|z_I\rangle e^{-z^*_{F\alpha} z_{F\alpha}}\,. \tag{D.6}$$

Here $\mathcal{V}$ is the on-site Coulomb term and the source term appearing in (3.3)–(3.5), and $\mathcal{V}(a^\dagger_\gamma a_\gamma)$, z_I and z_F are abbreviations of $\mathcal{V}(\{a^\dagger_\gamma\}, \{a_\gamma\})$, $\{z_{I\gamma}\}$ and $\{z_{F\gamma}\}$ respectively. As usual we divide the exponential into $N+1$ pieces and insert N multimode complete sets like (D.4). We get

$$\left(\prod_{i=1}^{N}\prod_{\alpha}\int dz^*_{i\alpha}dz_{i\alpha}\right) e^{-\sum_{i=1}^{N+1} z^*_{i\alpha}z_{i\alpha}} e^{-\sum_{i=1}^{N+1} z^*_{i\alpha}z_{i-1\alpha}}$$
$$\times\, e^{-\varepsilon\sum_{i=1}^{N+1}\{t_{\alpha\beta}z^*_{i\alpha}z_{i-1\beta} + \mathcal{V}(z^*_{i\gamma}z_{i-1\gamma})\}}\,, \tag{D.7}$$

where $\varepsilon = \beta/(N+1)$, $z_{0\alpha} = z_{I\alpha}$, $z_{N+1\alpha} = z_{F\alpha}$ and we have assumed that $\mathcal{V}$ is normal-ordered. The first two exponentials can be *formally* written as

$$e^{-\varepsilon\sum_{i=1}^{N+1} z^*_{i\alpha}(z_{i\alpha} - z_{i-1\alpha})/\varepsilon} \to e^{-\int_0^\beta d\tau z^*_\alpha(\tau)\dot{z}_\alpha(\tau)}\,. \tag{D.8}$$

In this way, through the trace formula (D.5), we obtain the path integral representation of (3.1), arriving at (3.11).

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