Unified time-path approach to the generating functional of the Brownian oscillator system: The bilinearly corrected Feynman rule for nonequilibrium processes

K. Okumura and Y. Tanimura
Division of Theoretical Studies, Institute for Molecular Science, Myodaiji, Okazaki, Aichi 444, Japan
(Received 26 May 1995)

We derive fully corrected propagators of a bilinearly interacting Brownian oscillator system by summing up an infinite number of Feynman diagrams. The generating functional of a nonequilibrium system is calculated in terms of the bilinearly corrected propagators thus obtained. The result offers the Feynman rule for systematically studying the effects of both anharmonicity of the potential and nonbilinear system-bath couplings on the Brownian oscillator system. The reduced density matrix, which is useful for investigating the dynamics of the system, is also calculated. The unified time path, which is powerful in calculating propagators, is introduced and used effectively throughout the work.

PACS number(s): 05.40.+j, 05.70.Ln, 42.65.—k, 82.20.—w

I. INTRODUCTION

Feynman rule for a nonequilibrium quantum system has been widely used effectively in various fields. The temperature Green function was introduced, from which the information on the real-time process was pulled out by using a rather intricate method of analytical continuation [1–4]. On the other hand, the double-path method starts from the real-time instead of the imaginary time [5–8].

Recently the double-path formalism is extended to a formalism in which the propagator becomes a $3 \times 3$ matrix instead of a $2 \times 2$ of the double-path formalism [9,10]. Typical nonequilibrium processes are explicitly calculated through the Feynman rule in terms of the $3 \times 3$ propagators [11]. We introduce the unified time path from this formalism in the following section. The path integral is directly performed on this unified time path.

In this paper we consider a nonequilibrium process where the total system is initially in the equilibrium state and then the time-dependent external force (applied after the initial time) derives the total system towards a nonequilibrium state. In such a case the unified time-path approach gives a unified perspective to calculation and argument as we see below. The quantity explicitly derived below is the generating functional for this case of nonequilibrium process, from which the Feynman rule is readily known and thus all the physical quantities can be estimated by diagrams. The generating functional is here defined as a functional of the external force which is obtained from the density matrix by tracing over all degrees of freedom of the total system [see Eq. (2.11) below]. The generating functional of nonequilibrium system provides a convenient means for incorporating effects of dissipation in physical systems [12].

We intend to calculate the generating functional of the Brownian motion model. One of the most fruitful models of the Brownian motion is described by the system-bath Hamiltonian [13],

$$H_{SB} = \frac{p^2}{2M} + \frac{M\Omega^2}{2} Q^2 + \sum_{i=1}^{N} \left( \frac{p_i^2}{2m_i} + \frac{m_i\omega_i^2}{2} q_i^2 \right) - Q \sum_{i=1}^{N} c_i q_i + V(Q,q,t). \tag{1.1}$$

Following the work by Caldeira and Leggett [14], there has been a renewed interest in the problem of Brownian motion. This model has been successfully used to study problems in chemical reaction [15,16], electron transportation in the semiconductor [17], and quantum optics [18]. Various techniques have been developed to include the effects of $V(Q,q,t) \neq 0$ both numerically and analytically [19–23].

Though many studies have been done by using this system-bath Hamiltonian in diverse fields, there were no systematic and analytical ways of calculating the effects of the anharmonicity of the potential and the nonbilinear coupling [i.e., the case of $V(Q,q,t) \neq 0$]. [Though the interaction $Q \sum_{i=1}^{N} c_i q_i$ is usually called linear, we call it bilinear or linear-linear in what follows since, if we view the system (1.1) as a special case of (2.1) with (2.2) below, it may be appropriate to call the term $\sum_{i,j=1}^{N} c_i q_i q_j \rightarrow Q \sum_{i=1}^{N} c_i q_i$ bilinear interacting term.] This is partly because the Feynman rule in which propagators are fully corrected by the bilinear interaction has not been derived up to now. Needless to say, this line of study is important since the real mechanism of dissipation may be much more complex than the simple harmonic case.

In this paper we derive the bilinearly corrected $3 \times 3$ propagators for a general bilinear coupling term $\sum_{i,j=1}^{N} c_i q_i q_j$ and obtain an expression for the generating functional in terms of these propagators [see Eq. (6.1) below]. This expression gives a Feynman rule for a nonequilibrium system where anharmonic interactions correspond to vertices. Note here that the bilinear interaction does not appear as vertices since it is fully taken into the propagator. Thus we can systematically study
the nonequilibrium process of the total system not only for the case of anharmonic potential (both for the system and for the bath) but also for the case of general (nonbilinear) mechanism of the system-bath interactions. We also show that the reduced density matrix, which is useful for studying the dynamics of the system, can be calculated from the generating functional.

Finally we point out the close relation between [24] and the present work. In [24] they derived various correlation functions for the case of the harmonic system potential and the bilinear system-bath coupling. Some of their expressions can be regarded as elements of our $3 \times 3$ propagators for the special coupling case [see Eq. (4.19)] while they did not derive the other elements of the propagator even for this special coupling case. The general coupling case is not dealt with in [24] at all. In this sense the present work can be viewed as a generalization of [24]. We show explicitly, however, how these elements of propagators can be used as the basis for investigating the effect of the anharmonicity or nonbilinear coupling by using the Feynman diagrammatic technique. Especially, the nonbilinear coupling correction can be systematically examined in our formulation.

II. THE UNIFIED TIME PATH

Let us consider a quantum system of $N + 1$ degrees of freedom. The total Hamiltonian is expressed in a generalized form as

$$H_T(p, q) = \sum_{i=0}^{N} \left( \frac{p_i^2}{2m_i} + \frac{m_i}{2} \frac{\hat{q}_i^2}{q_i^2} \right) + V_T(q, t),$$

(2.1)

$$V_T(q, t) = -\frac{1}{2} \sum_{i,j=0}^{N} c_{ij} q_i q_j + V(q, t),$$

(2.2)

where $V(q, t)$ is the nontrivial part of the interaction [for example, $V(q, t) = \lambda q_0^2 \sum_{i=1}^{N} q_i$]. The conventional system-bath Hamiltonian can always be cast into this form, if we set $\{p_0, q_0, \omega_0, m_0\} \equiv (P, Q, \Omega, M)$ and regard the other degrees of freedom as the bath degrees. Strictly speaking, the bath degrees of freedom $N$ should be taken to infinity at some stage in order to dissipate the energy of the system $(P, Q)$ to the bath.

We are interested in the case where the total system is initially in equilibrium with the temperature $1/\beta$, and then the time-dependent external force is applied at some time, say $t = 0$. The external force brings the total system to a nonequilibrium state. In this context the total nontrivial interaction $V(q, t)$ is really time dependent while initially it is only a function of the coordinates, i.e., $V_I(q) \equiv V(q, 0)$. In what follows we assume $c_{ij} = c_{ji}$ time independent without loss of generality.

The experimental observable of the system is always expressed by the expectation value of various operators or correlation functions. Such quantities can be systematically investigated when the generating functional is introduced. To this end, we add artificial source terms to the original Hamiltonian,

$$H(t) = H_T(p, q) \rightarrow H^{J_{\alpha}}(t)$$

$$= H(t) - \sum_i J_{\alpha,i}(t) q_i \quad (\alpha = 1, 2).$$

(2.3)

Then the time evolution operators from initial time to infinite future (and vice versa) of the total system with the source are given by

$$K^{J_{\alpha}} = T e^{-\frac{1}{\hbar} \int_0^\infty dt H^{J_{\alpha}}(t)}$$

$$= \tilde{T} e^{-\frac{1}{\hbar} \int_0^\infty dt H^{J_{\alpha}}(t)},$$

(2.4)

where $T$ and $\tilde{T}$ are the usual time ordering and antitime ordering operators, respectively. We introduce the generalized version of the unnormalized initial density matrix,

$$\sigma^J_{\alpha} = T e^{-\frac{1}{\hbar} \int_0^\infty d\tau H^{J_{\alpha}}(\tau)},$$

(2.5)

where $T_\alpha$ is the $\tau$-ordering operator and $H^{J_{\alpha}}(\tau)$ is given by

$$H^{J_{\alpha}}(\tau) = H(t = 0) - \sum_i \dot{J}_{\alpha,i}(\tau) q_i.$$  

(2.6)

Then the normalized initial equilibrium density matrix of the original system is expressed as

$$\rho_I = \left( \sigma^J_{\alpha} / Tr \sigma^J_{\alpha} \right)_{J_{\alpha} = 0}.$$  

(2.7)

Consider the generating functional for the correlation function $Z_J$ or that for the connected correlation function $W_J$ defined by

$$Z_J = e^{W_J} = Tr \left( \sigma^J_{\alpha} \tilde{K}^{J_{\alpha}} K^{J_{\alpha}} \right).$$  

(2.8)

By using the generating functional, the expectation value of the coordinate of the system at $t$ is given as

$$\langle \dot{q}_i \rangle_t = \left. \frac{\partial W}{\partial J_{\alpha,i}(t)} \right|_{J_{\alpha} = 0} = -\left. \frac{\partial W}{\partial J_{\alpha,i}(t)} \right|_{J_{\alpha} = 0} \quad (t \geq 0),$$  

(2.9)

$$\langle \dot{q}_i \rangle_{t=0} = \frac{1}{i} \left. \frac{\partial W}{\partial \dot{J}_{\alpha,i}(\tau)} \right|_{\tau=0}. \quad (2.10)$$

Here, $J = 0$ implies $J_{\alpha,i} = \dot{J}_{\alpha,i} = \ddot{J}_{\alpha,i} = 0$ for all $i$. We can also obtain the expectation value of the momentum or the correlation function including both $p_i$ and $q_i$ [after replacing all $p_i(t)$ with $m_i q_i(t)$] by taking derivatives of $W$ with respect to $J_1$ [10,25] [see also Eq. (5.20)].

At this point we introduce the unified time path $C = C_1 + C_2 + C_3$ in the complex time plane as in shown Fig. 1 (see Ref. [10]). Namely, it starts from the origin up to infinity along the real path $(C_1)$, returns to the origin $(C_2)$, and then goes to $-i \beta \hbar$ along the imaginary axis $(C_3)$.

The step function $\theta_C(t, t')$ on $C$ is naturally defined such that $\theta_C(t, t')$ takes the value unity if the time $t$ is later than the time $t'$; otherwise $\theta_C(t, t') = 0$. The word later here means that the time $t$ appears later than the
Z = \int [q] e^{i \int_{C} dt \mathcal{L}(t)} , \quad (2.13)

where \( \int [q] \) implies the path integration on the unified time path and \( \mathcal{L}(t) \) is the Lagrangian corresponding to \( H(t) \) (see Appendix A). Now that all the quantities appearing in \( Z \) become the \( c \) number, we can pull out the interaction term by using the simple identity

\[
e^{-i \int_{C} dt V_{T}(s) \mathcal{L}(t)} = e^{\mathcal{L}(t) e^{-i \int_{C} dt V_{T}(s) \mathcal{L}(t)}},
\]

where

\[
Z_{s,i}^{(0)} = \int [q] e^{i \int_{C} dt \left[ \frac{i}{2} \frac{\partial}{\partial t} q_{i}^{2}(t) - \frac{m_{i}^{2}}{2} q_{i}^{2}(t) + J_{s}(t) q_{i}(t) \right]} , \quad (2.15)
\]

and \( V_{T}(s) \) implies \( V_{T}(q, t) \) with \( q_{i} \) replaced by \( \frac{\partial}{\partial q_{i}(t)} \).

Although the quantity \( Z_{s,i}^{(0)} \) itself has been already obtained \([24, 10]\), we present a quick derivation based on the unified time path. For simplicity we drop the index \( i \) of \( q_{i} \). The well-known result (see, for example, \([26]\)) on the \( C_{1} \) path,

\[
\langle q' | T e^{-i \int_{0}^{T} dt \left( \frac{\partial^{2}}{2m} q_{i}^{2}(t) - J_{s}(t) q_{i}(t) \right)} | q \rangle = \left( \frac{m \omega}{2 \pi i \hbar \sin \omega T} \right)^{1/2} e^{i S}, \quad (2.16)
\]

\[
S = \frac{m \omega}{2 \sin \omega T} [(q^{2} + q'^{2}) \cos \omega T - 2qq'] + \frac{1}{\sin \omega T} \int_{0}^{T} dt J_{s}(t) [q \sin \omega (T - t) + q' \sin \omega t] + \frac{1}{2} \int_{0}^{T} dt \int_{0}^{T} ds J_{s}(t) G_{11}(t - s) J_{s}(s), \quad (2.17)
\]

\[
G_{11}(t - s) = - \frac{1}{m \omega} \left[ \theta(t - s) \frac{\sin \omega (T - t) \sin \omega s}{\sin \omega T} + (t \leftrightarrow s) \right], \quad (2.18)
\]

is generalized on the unified path \( C \) as

\[
\langle q' | T e^{-i \int_{C} dt \left[ \frac{i}{2} \frac{\partial}{\partial t} q_{i}^{2}(t) - \frac{m_{i}^{2}}{2} q_{i}^{2}(t) + J_{s}(t) q_{i}(t) \right]} | q \rangle = \left( \frac{m \omega}{2 \pi \hbar \sinh \omega \hbar} \right)^{1/2} e^{i S_{C}}, \quad (2.19)
\]

where \( S_{C} \) is given by \( S \) with the following replacements:
\[ T \rightarrow -i\beta \hbar, \]
\[ \int_0^T dt \rightarrow \int_C dt, \]
\[ J_1(t) \rightarrow J(t) = J_\alpha(t) \quad (t \in C_\alpha), \]
\[ \theta(t-s) \rightarrow \theta_C(t,s). \]  

(2.20)

We emphasize the fact that, although these replacements may be almost self-evident, the above result can be directly obtained by natural extension of the known methods (see Appendix A). On the contrary, they first derive \( \langle q|\sigma^J|q'\rangle, \langle q'|K^J_2|q''\rangle, \langle q''|K^J_1|q\rangle \) separately and then performed integration with respect to \( q, q', q'' \) to derive the quantity \( \text{Tr} \sigma^J K^J_2 K^J_1 \) in [24,10].

The generating functional \( Z_J^{(0)} \) is obtained from Eq. (2.19) by tracing over the coordinate, i.e., by setting \( q = q' \) and integrating over \( q \). We thus get the known result,

\[ Z_J^{(0)} = e^{iW_J^{(0)}} = \frac{1}{2\sinh(\omega \beta \hbar/2)} e^{\Phi_J}, \]

(2.21)

\[ \Phi_J = \frac{1}{2} \left( \frac{i}{\hbar} \right)^2 \int_C dt \int_C ds J(t)D(t-s)J(s), \]

(2.22)

\[ D(t-s) = \frac{\hbar}{2m \omega \sinh(\omega \beta \hbar/2)} \]

\[ \times [\theta_C(t-s) \cos(\omega(t-s - i\beta \hbar/2) + t \leftrightarrow s)]. \]

(2.23)

Thus the notion of the unified time path which was introduced in Ref. [10] can be extended in a useful way.

By recovering the index \( i \) of \( q_i \), we arrive at the following expression for the generating functional for the total system:

\[ Z_J = e^{-i\int_C dt V(s \tilde{J}(t))} Z_J^L, \]

(2.24)

\[ Z_J^L = e^{iW_J^L} = e^{i\int_C dt \sum_{i,j} \tilde{s}_{ij}(t) \partial \tilde{s}_{ij}(t)} e^{\frac{1}{2} \left( \frac{i}{\hbar} \right)^2 \int_C dt \int_C ds \sum_i J_i(t) D_i(t-s) J_i(s)}, \]

(2.25)

where \( D_i(t-s) \) is defined in Eq. (2.23) with \( m, \omega \) replaced by \( m_i, \omega_i \), and \( V(\Phi J) \) implies \( V(q,t) \) in Eq. (2.2) with \( q_i \) replaced by \( \frac{\hbar}{2} \frac{\partial}{\partial J_i(t)} \).

By using the trivial identity (see Appendix B)

\[ e^{-\frac{1}{2} \tilde{s}_{ij}(t) A_i(t) s_{ij}(t) B_j(t)} = e^{\frac{1}{2} J_i(t) A_i(t) J_i(t) B_i(t)}}, \]

(2.26)

which holds for matrices \( A, B \) and the vector \( J \), we see that \( Z_J^L \) can be formally cast into the following form:

\[ Z_J^L = e^{iW_J^L}, \]

(2.27)

\[ \frac{i}{\hbar} W_J^L = \frac{1}{2} \left( \frac{i}{\hbar} \right)^2 \int_C dt \int_C ds \sum_{i,j} J_i(t) D_{ij}(t-s) J_j(s), \]

(2.28)

where

\[ [D_{ij}(t-s)]^{-1} = [D_{ij}(t-s) \delta_{ij}]^{-1} - \frac{i}{\hbar} c_{ij} \delta(t-s), \]

(2.29)

and \( \delta_{ij} \) is Kronecker's delta. The inverse in the above expression is the inverse of the matrix when you consider \( D_{ij}(t-s) \) as the matrix \( D_{ab} \) with \( a = (i,t), b = (j,s) \). Moreover note here that \( t, s \) is on the unified time path. In Sec. IV, we derive the explicit form of the bilinearly corrected propagator \( D_{ij}(t-s) \) by noting the graphical meaning of Eq. (2.26), which will become clear as we go along. It should be emphasized that Eq. (2.24) with (2.27) and (2.28) gives the Feynman rule based on the bilinearly corrected propagator. This Feynman rule is a starting point of studying a nonequilibrium system subject to an anharmonic potential in addition to the linear-linear interaction.

III. THE LAPLACE-FOURIER REPRESENTATION

If we introduce variables \( J^{(+)} \) and \( J^{(-)} \) by

\[ J^{(+)} = (J_1 + J_2)/2, \quad J^{(-)} = J_1 - J_2, \]

(3.1)

then the quantity \( \Phi_J \) in Eq. (2.22) can be expressed as [24,10]

\[ \Phi_J = \left( \frac{i}{\hbar} \right)^2 \int_0^\infty dt \int_0^\infty ds \left( J^{(-)}(t) D_0^{(-)}(t-s) J^{(+)}(s) + \frac{1}{2} J^{(-)}(t) D_0^{(-)}(t-s) J^{(-)}(s) \right) \]

\[ + \frac{i}{\hbar} \int_0^\infty dt \int_0^{\beta \hbar} d\tau J^{(-)}(t) D_0^{(-)}(t+i\tau) J_3(\tau) + \frac{1}{2} \frac{1}{\hbar^2} \int_0^{\beta \hbar} d\tau \int_0^{\beta \hbar} d\tau' J_3(\tau) D_0^{(3)}(-i\tau + i\tau') J_3(\tau'), \]

(3.2)
where
\[ D_{0}^{-,+}(t) = \frac{\hbar}{i} \theta(t) \frac{\sin \omega t}{m \omega}, \]
\[ D_{0}^{-,-}(t) = \frac{\hbar}{2m \omega} \cos \omega t, \]
\[ D_{0}^{-,-}(t + i\tau) = \frac{\hbar}{2m \omega} \frac{\cos \omega (t + \beta \hbar/2)}{\sinh(\omega \beta \hbar/2)}, \]
\[ D_{0}^{(33)}(-i\tau) = \frac{\hbar}{2m \omega} \frac{\cos \omega (\tau + \beta \hbar/2)}{\sinh(\omega \beta \hbar/2)} + (\tau \leftrightarrow -\tau). \]
(3.3)

By noting that
\[ D_{0}^{-,+}(t-s) = \frac{\hbar}{i} \frac{\partial}{\partial f_{J}^{-,+}(t)} \frac{\partial}{\partial f_{J}^{(+)}} W_{J}(0) \bigg|_{J=0}, \]
(3.4)
with \( W_{J}(0) \) defined in Eq. (2.22), we have
\[ D_{0}^{-,+}(t-s) = \theta(t-s)[\langle q(t-s)q \rangle_{0} - \langle qq(t-s) \rangle_{0}], \]
(3.5)
where \( \langle X \rangle_{0} \) is the connected part of the average of \( X \) by the initial density matrix for the noninteracting system. The time evolution is also done by the noninteracting Hamiltonian. Explicitly,
\[ \langle q(t)q \rangle_{0} = \langle q(t)q \rangle_{0} - \langle q \rangle^{2}_{0}, \]
(3.6)
where
\[ \langle q(t)q \rangle_{0} = \text{Tr} \left( e^{-\beta H} e^{i\mathcal{H}_{1}/\hbar} q e^{-i\mathcal{H}_{1}/\hbar} \right) / \text{Tr} e^{-\beta H}, \]
(3.7)
with \( H = p^{2}/2m + m \omega^{2}q^{2}/2 \). Similarly we have
\[ D_{0}^{-,-}(t-s) = \frac{i}{2} \langle q(t-s)q + qq(t-s) \rangle_{0}. \]
(3.8)
Since \( \langle q(t-s)q \rangle_{0} \) is the complex conjugate of \( \langle qq(t-s) \rangle_{0} \), we can recast the above relation into
\[ D_{0}^{-,+}(t) = -2i\theta(t) \text{Im} \langle qq(t) \rangle_{0}, \]
\[ D_{0}^{-,-}(t) = \text{Re} \langle qq(t) \rangle_{0}. \]
(3.9)
(3.10)

We also have
\[ D_{0}^{-,-}(t + i\tau) = \langle qq(t + i\tau) \rangle_{0}, \]
\[ D_{0}^{(33)}(-i\tau) = \theta(\tau) \langle qq(-i\tau) \rangle_{0} + (\tau \leftrightarrow -\tau). \]
(3.11)
(3.12)

Equations (3.9)–(3.12) imply the fact that once the quantity \( D_{0}^{-,-}(t + i\tau) \) is evaluated the others can be obtained as a special value of this quantity. For example, \( D_{0}^{(33)}(-i\tau) \) is given by \( D_{0}^{-,-}(t + i\tau) \) evaluated at \( t = 0, \tau' = -\tau \) [see Eq. (3.3)] if \( \tau > 0 \).

We note here that these equations also imply the well-known fact that one can obtain all the expressions by the analytic continuation of the Matsubara Green function \( D_{0}^{(33)}(-i\tau) \).

By introducing the Laplace-Fourier transform
\[ X(z, n) \equiv \int_{0}^{\beta \hbar} d\tau e^{-i\nu_{n}\tau} \int_{0}^{\infty} dt e^{-zt} X(t, \tau) \]
(0 < \( t, 0 < \tau < \beta \hbar), \]
we have
\[ D_{0}^{-,+}(t) = \int_{C_{+}} \frac{dz}{2\pi i} e^{zt} D_{0}^{-,+}(z) \quad (t > 0), \]
\[ D_{0}^{-,-}(t) = \int_{C_{-}} \frac{dz}{2\pi i} [\theta(t) e^{zt} + \theta(-t) e^{-zt}] D_{0}^{-,-}(z), \]
(3.15)
\[ D_{0}^{-,-}(t + i\tau) = \frac{1}{\beta \hbar} \sum_{n=-\infty}^{\infty} e^{i\nu_{n} \tau} \int_{C_{+}} \frac{dz}{2\pi i} e^{zt} D_{0}^{-,-}(z, n) \]
(0 < \( t), \]
(3.16)
\[ D_{0}^{(33)}(-i\tau) = \frac{1}{\beta \hbar} \sum_{n=-\infty}^{\infty} e^{i\nu_{n} \tau} D_{0}^{(33)}(n) \]
(0 < \( \tau < \beta \hbar), \]
(3.17)
where
\[ D_{0}^{-,+}(z) = \frac{\hbar}{i} f(z), \]
(3.18)
\[ D_{0}^{-,-}(z) = -\frac{\hbar}{\beta \hbar} \sum_{n=-\infty}^{\infty} \frac{z}{z^{2} - \nu_{n}^{2}} [f(z) - f(\nu_{n})], \]
(3.19)
\[ D_{0}^{-,-}(z, n) = -\frac{\hbar}{\beta \hbar} \frac{z + \nu_{n}}{z^{2} - \nu_{n}^{2}} [f(z) - f(\nu_{n})], \]
(3.20)
\[ D_{0}^{(33)}(n) = \hbar f(\nu_{n}). \]
(3.21)

Here, \( \nu_{n} \) is the Matsubara frequency defined by
\[ \nu_{n} = 2\pi n/\beta \hbar, \]
(3.22)
and the function \( f(z) \) by
\[ f(z) = \frac{1}{m \omega^{2} + z^{2}}. \]
(3.23)

The contour \( C_{\pm} \) on the complex \( z \) plane runs parallel to the imaginary axis where the real part is chosen so that there are no poles on the left side of the path. Note that \( D_{0}^{-,-}(z) \) cannot be denoted as the summation of \( D_{0}^{-,-}(z, n) \) since the term \( \frac{z + \nu_{n}}{z^{2} - \nu_{n}^{2}} \) is odd in terms of \( n, \) is singular in the sense that it behaves like \( 1/n \) as \( n \to \infty. \)

Equations (3.14)–(3.17) with (3.18)–(3.21) may be easily checked if we use \( 0 < \tau < \beta \hbar) \)
\[
cosh \omega (\tau - \beta t/2) = \frac{1}{\beta \hbar} \sum_{n=-\infty}^{\infty} \frac{\omega e^{i\nu_n \tau}}{\omega^2 + \nu_n^2}.
\]
(3.24)

\[
sinh \omega (\tau - \beta t/2) = \frac{1}{\beta \hbar} \sum_{n=-\infty}^{\infty} \frac{i\nu_n e^{i\nu_n \tau}}{\omega^2 + \nu_n^2}.
\]
(3.25)

and

\[
\frac{1}{z^2 - \nu_n^2} [f(z) - f(\nu_n)] = -mf(z) f(\nu_n).
\]
(3.26)

Note that the summand in the right-hand side of Eq. (3.25) is singular implying jumps at \( \tau = 0, \beta t \). This fact leads to the following relation to be used later:

\[
\lim_{\tau \to 0} \frac{1}{\beta \hbar} \sum_{n=-\infty}^{\infty} \frac{e^{i\nu_n \tau}}{i\nu_n} = \pm \frac{1}{2}.
\]
(3.27)

The prime means that the summation excludes the contribution from \( n = 0 \).

By virtue of Eqs. (3.9)–(3.12), once one knows \( D_0^{(-3)}(z, n) \), one can derive the other propagators. Though the process is trivial, it is useful when we derive the bilinearly corrected propagators so that we examine this process. Let us start from

\[
D_0^{(-3)}(t + \tau) = \frac{-\hbar}{\beta \hbar} \sum_{n=-\infty}^{\infty} e^{i\nu_n \tau} \int_{C_z} \frac{dz}{2\pi i} e^{zt} \frac{z + \nu_n}{z^2 - \nu_n^2} [f(z) - f(\nu_n)].
\]
(3.28)

If we use

\[
\int_{C_z} \frac{dz}{2\pi i} \frac{z + \nu_n}{z^2 - \nu_n^2} [f(z) - f(\nu_n)] \to -f(\nu_n) \quad (t \to 0^+),
\]
(3.29)

which is clear from the initial value theorem of the Laplace transformation, we get the following result by setting \( t = 0 \) in Eq. (3.28):

\[
D_0^{(3)}(-i\tau) = \frac{\hbar}{\beta \hbar} \sum_{n=-\infty}^{\infty} e^{i\nu_n \tau} f(\nu_n).
\]
(3.30)

Thus we obtain \( D_0^{(3)}(n) = hf(\nu_n) \) as desired. Next, by noting a simple relation (for \( n \neq 0 \))

\[
\frac{z + \nu_n}{z^2 - \nu_n^2} = \frac{z}{z^2 - \nu_n^2} + \frac{1}{\nu_n} \frac{z^2}{z^2 - \nu_n^2} - \frac{1}{\nu_n},
\]
(3.31)

we have

\[
\frac{1}{\beta \hbar} \sum_{n=-\infty}^{\infty} e^{i\nu_n \tau} \frac{z + \nu_n}{z^2 - \nu_n^2} [f(z) - f(\nu_n)]
\]
(3.32)

\[
\to \frac{1}{\beta \hbar} \sum_{n=-\infty}^{\infty} \frac{z}{z^2 - \nu_n^2} [f(z) - f(\nu_n)]
\]

\[
- \frac{i}{2} f(z) \quad (\tau \to 0^+).
\]
(3.33)

We have used Eq. (3.27) and the fact that nonsingular terms, which are odd in terms of \( n \), are summed up to zero. If we set \( \tau = 0^+ \) in Eq. (3.28), the left-hand side becomes \( \langle \Phi q(t) \rangle \). Since \( D_0^{(+)} \) and \( D_0^{(-)} \) are the imaginary and real part of \( \langle \Phi q(t) \rangle \), respectively, we get Eqs. (3.18) and (3.19) as desired.

**IV. DERIVATION OF THE BILINEARLY CORRECTED PROPAGATORS**

With the \( J^{(\pm)} \) representation, we can recast Eq. (2.25) into the following form:

\[
Z^L_j = e^{iW^L_j} = e^{\sum_i \Phi_{ij}(\hbar_b) e^{\sum_i \Phi_{ij}} \equiv e^{\Phi^+}},
\]
(4.1)

where

\[
\Psi_{ij} \left( \frac{\partial}{\partial J^+} \right) = \frac{i}{\hbar} \left( \frac{h}{i} \right)^2 c_{ij} \int_0^\infty dt \left( \frac{\partial}{\partial J_{i+}(t)} \frac{\partial}{\partial J_{j-}(t)} \right)
\]

\[
+ \frac{1}{\hbar} \left( \frac{h}{i} \right)^2 c_{ij} \int_0^\infty dt \frac{\partial}{\partial J_{3i}(t)} \frac{\partial}{\partial J_{3j}(t)}
\]
(4.2)

and \( \Phi_{J_{i+}}, \Phi^+ \) are given by making the following replacement in Eq. (3.2) [see also Eq. (6.2)].

Replacements with respect to \( \Phi_{J_{i+}} \):

\[
J^{(-)}(J^{(+)}, J_3) \to (J^{(-)}, J^{(+)}_3), (J^{(-)}, J_3),
\]
(4.3)

\[
(D^{(-)}, D^{(-)}_3, D^{(-)}_0, D^{(3)}_0) \to (D^{(-)}_0, D^{(3)}_0, D^{(-)}_0, D^{(3)}_0)
\]
(4.4)

where \( D^{(l)}_0 \) is given by the replacement:

\[
(m, \omega) \to (m, \omega_i)
\]
(4.5)

in Eq. (3.3). In the Laplace-Fourier representations in Eqs. (3.18)–(3.21), \( D^{(l)}_0 \) is given through the following replacement:

\[
f(x) = \frac{1}{m} \frac{1}{\omega^2 + x^2} \to f_i(x) = \frac{1}{m_i} \frac{1}{\omega_i^2 + x^2}.
\]
(4.6)

Replacements with respect to \( \Phi^+ \) [see (6.2)]:

\[
(J^{(-)}, J^{(+)}, J_3) \to (J^{(-)}, J^{(+)}_3), (J^{(-)}, J_3),
\]
(4.7)

\[
(D^{(-)}, D^{(-)}_3, D^{(-)}_0, D^{(3)}_0)
\]

\[
\to (D^{(-)}, D^{(-)}_3, D^{(-)}_0, D^{(3)}_0).
\]
(4.8)
These quantities $D_{ij}^{(t_m)}$ are the bilinearly corrected propagators we aim at deriving in what follows.

Let us begin with the derivation of $D_{ij}^{(-+) \rightarrow}$. This quantity is expressed as

$$D_{ij}^{(-+)}(t) = \frac{\hbar}{i} \theta(t) \frac{\partial}{\partial J_i^{(-)}(t)} \frac{\partial}{\partial J_j^{(+)}(0)} W_L^J$$

$$= \theta(t) \langle q_i(t) q_j - q_j q_i(t) \rangle,$$  \hspace{1cm} (4.9)

where $\langle X \rangle$ is the average of $X$ by the initial bilinearly corrected density matrix whose dynamics is governed by the bilinearly corrected Hamiltonian. For example, $\langle q_i(t) q_j \rangle$ is given by Eq. (3.6) with Eq. (3.7) under the replacement

$$q \rightarrow q_i, \quad H \rightarrow \sum_i \left( \frac{q_i^2}{2m_i} + \frac{m_i \omega_i^2}{2} q_i^2 \right) - \sum_{ij} q_i c_{ij} q_j.$$  \hspace{1cm} (4.10)

Thus we can calculate $D_{ij}^{(-+)}(t)$ by using the following Feynman rule. Note that this rule is immediately obtained from the third expression in Eq. (4.1).

\[
\begin{array}{cccc}
\text{Propagators} & \text{Vertices} & \text{External points} \\
\hline
- & - & + & D_{ij}^2 \\
- & i & - & D_{ij}^3 \\
- & i & 3 & D_{ij}^3 \\
- & i & 3 & D_{ij}^3 \\
\end{array}
\]

By using the fact that $D_{ij}^{(-+) \rightarrow}(t)$ is causal, we easily find the following graphical expression:

$$D_{ij}^{(-+)}(t) = \cdots$$  \hspace{1cm} (4.11)

Let us consider the third graph explicitly. The algebraic expression of this quantity is given by

$$\left( \frac{i}{\hbar} \right)^3 \int_0^t dt' \int_0^{t'} dt'' D_{0,i}^{(-+)}(t-t') c_{ik} D_{0,k}^{(-+)}(t'-t'') c_{kj} D_{0,j}^{(-+)}(t'')$$

$$= \frac{\hbar}{i} \int_C \frac{dz}{2\pi i} f_i(z) c_{ik} f_k(z) c_{kj} f_j(z).$$  \hspace{1cm} (4.12)

Thus if we define matrices

$$[F_d(z)]_{ij} = f_i(z) \delta_{ij},$$

$$[F(z)]_{ij} = C_{ij},$$  \hspace{1cm} (4.14)

we have

$$D_{ij}^{(-+)}(z) = \frac{\hbar}{i} \tilde{F}_{ij}(z).$$  \hspace{1cm} (4.16)

Here we have introduced the matrix $\tilde{F}(z)$ by

$$[\tilde{F}(z)]^{-1} = [F_d(z)]^{-1} - F(z),$$  \hspace{1cm} (4.17)

where the inverse implies the one regarding $(i,j)$ as the indices of the matrices.

We can calculate $D_{ij}^{(-+) \rightarrow}(t)$ in a similar way. The graphs at the second order in $c_i$ is given in Fig. 2. Due to the causality the number of graphs is considerably re-
UNIFIED TIME-PATH APPROACH TO THE GENERATING . . .

For the case where the linear-linear coupling matrix \( c_{ij} \) takes a nonzero value only when
\[
c_{0j} = c_{j0} \equiv c_j \quad (j \neq 0),
\]
the expressions of the diagonal elements \( D_{00}^{(\pm)}(z) \) and \( D_{00}^{(-)}(z) \) have been obtained in Ref. [24] [expressed as \( 2i\hat{A}(z) \) and \( \hat{S}(x) \), respectively]. In this case the matrix \( \hat{F}(x) \) is replaced by a mere number \([1/fo(x) - fB(x)]^{-1}\) with \( fB(x) = \sum_{i=1}^{N} c_i^2 f_i(x) \). Their \( \hat{\gamma}(x) \) and \( \mu \) satisfy \( Mx\hat{\gamma}(x) = -fB(x) \) with \( M \equiv m_0 \). If we recall that they started from the system Hamiltonian \( H^{GSI} = P^2/2M + (M\Omega^2 + \mu)Q^2/2 \), we find that our results reduce to theirs.

The Fourier transformation for the real-time variables instead of the Laplace transformation (the Fourier-Fourier representation) is easily obtained if we make the replacement
\[
\int_{C_i} \frac{dz}{2\pi i} \rightarrow \int_{-\infty}^{\infty} \frac{dk}{2\pi} \quad e^{zt} \rightarrow e^{ikt}
\]
in Eqs. (3.14)–(3.17) and
\[
z \rightarrow i(k - i0^+)
\]
in Eqs. (3.18)–(3.21), etc. In this representation we can also show easily that our results reduce to those in Ref. [24] for the above special case of Eq. (4.19).

### V. THE REDUCED DENSITY MATRIX

The coordinate representation of the reduced density matrix \( \rho_J(Q',Q,t) \) can be obtained by using the method similar to the one used in deriving the generating functional. In this section we regard the 0th coordinate and momentum as the system degrees of freedom, i.e., \( (p_0, q_0) = (P,Q) \). The reduced density matrix with the source \( J = J_{\alpha,i} (\alpha = 1,2,3; i = 1,\ldots,N) \) is given by
\[
\rho_J(t) = \frac{\text{Tr}_B \sigma_J(t)}{\text{Tr} \sigma_J(t)},
\]
where
\[
\sigma_J(t) = T_C e^{-i \int_C dt' H'(t')},
\]
and \( \text{Tr}_B \) (Tr) implies the trace over all degrees of freedom of the bath (the total system). The time \( t = T \) of
\[ \sigma_J(Q', Q, t) = e^{-\frac{i}{\hbar} \int_0^t \sum_{q} \frac{\partial^2}{\partial q^2} \sigma_J(r, x, t) \times \langle Q' | T_{cc} e^{-\frac{i}{\hbar} \int_0^t H_S(t') } | Q \rangle \times \prod_{i=1}^N e^{\Phi_{J,i}}. \] (5.5)

Here, \( \Phi_{J,i} \) is given in Eq. (4.1) and the second element in the product on the right-hand side is given by Eq. (2.19) with \((q', p, m, \omega, J, T)\) replaced by \((Q', Q, P, M, \Omega, J_0, t)\). The exponent of the second element in Eq. (5.5) is bilinear in terms of the vector \((Q', Q, J_0)\) [see Eq. (2.19)]. The functional \( \Phi_{J,i} \) is also bilinear in terms of the vector \((J_1, \ldots, J_N)\). Thus, we can write Eq. (5.5) in the following form:

\[ \sigma_J(Q', Q, t) = e^{-\frac{i}{\hbar} \int_0^t \sum_{q} \frac{\partial^2}{\partial q^2} \sigma_J(r, x, t) } \times (a \text{ factor independent of } Q', Q, J), \] (5.6)
If we note that
\[
\langle PQ \rangle = \int dx \frac{h}{i} \delta(x) \left( r \pm \frac{x}{2} \right) \rho_{ij}^2(r, x, t)
\]
we arrive at the result \( C = 0 \).

The equal-time autocorrelation of \( P \) becomes

\[
\sigma^2_j(r, x, t) = \sqrt{\frac{1}{2\pi(Q^2)}} e^{-\frac{(r - (Q^2))}{4(Q^2)} - (\frac{P^2}{2Q^2})x^2 + \frac{i}{2}(P)z + J \cdot DJ},
\]

where \( J \cdot DJ \) is given in Eq. (5.9). The quantities \( \langle Q \rangle, \langle P \rangle, \langle Q^2 \rangle, \langle P^2 \rangle \) are easily obtained through the generating functional and are given explicitly in terms of the bilinearly corrected propagators as follows.

\[
\langle Q \rangle = \frac{\partial W_j^2}{\partial J(-i)(t)} = \frac{i}{\hbar} \int_0^{t'} ds \sum_j \left[ D_{ij}^{(2)}(t - s)J_2^{(2)}(s) + D_{ij}^{(-2)}(t - s)J_2^{(-2)}(s) \right] + \frac{1}{\hbar} \int_0^{t'} d\tau \sum_j D_{ij}^{(-3)}(t + i\tau)J_3,(\tau),
\]

\[
\langle P \rangle = M \frac{\partial}{\partial i} \frac{\partial W_j^2}{\partial J(-i)(t)} = M \frac{\partial}{\partial i} \langle Q \rangle,
\]

\[
\langle Q^2 \rangle = \frac{\hbar}{i} \frac{\partial}{\partial J_0^{(-1)}(t)} \frac{\partial}{\partial J_0^{(-1)}(t')} W_j^2 \bigg|_{t = t' = 0} = D_0^{(-2)}(0),
\]

\[
\langle P^2 \rangle = M^2 \frac{\partial}{\partial i} \frac{\partial}{\partial i'} \frac{\hbar}{i} \frac{\partial}{\partial J_0^{(-1)}(t)} \frac{\partial}{\partial J_0^{(-1)}(t')} W_j^2 \bigg|_{t = t' = 0} = -M^2 D_0^{(-2)}(0).
\]

The above results generalize the expression for the density-matrix element given in Ref. [12] with considerably easier calculation. Namely, the external forces were restricted to \( F(t) \) and \( f(t) \) only [which correspond to \( J^-(-t) \) and \( J^+(t) \), respectively] in Ref. [12], while the three external forces \( J^-(-t), J^+(t), \) and \( J_3(t) \) are introduced in the present work. As a result, we can investigate anharmonicity and nonbilinear effects not only on the \( C_1 + C_2 \) path but also on the \( C_3 \) path. The density-matrix element enables us to study the differences between quantum and classical dynamics, particularly if we introduce the Wigner representation [12, 24].

### VI. CONCLUSION

We have derived the bilinearly corrected propagator by using a graphical technique. Thereby we have succeeded in taking the inverse in \((i, t)\) space with \( t \) on the unified time path [see below (2.29)]. The results are summarized as follows. The generating functional defined in (2.8) is given by

\[
Z_J = e^{-iL_W(s_i(t))} e^{iW_j^2},
\]

where the generating functional for the connected correlation function \( \frac{i}{2} W_j^2 \) given in Eq. (2.28) can be rewritten as

\[
\Phi^{+-} = \left( \frac{i}{\hbar} \right)^2 \int_0^\infty dt \int_0^\infty ds \left( J_1^{(-)}(t)D_2^{(2)}(t - s)J_2^{(2)}(s) + \frac{1}{2} J_1^{(-)}(t)D_2^{(-2)}(t - s)J_2^{(-2)}(s) \right) + \frac{i}{\hbar} \int_0^\infty dt \int_0^\infty ds \left( J_1^{(-)}(t)D_2^{(-2)}(t - s)J_2^{(-2)}(s) + \frac{1}{2} J_1^{(-)}(t)D_2^{(-2)}(t - s)J_2^{(-2)}(s) \right),
\]

where the Fourier-Laplace representations of the propagators are the following:
\[
D_{ij}^{(\pm)}(t) = \int_{C_z} \frac{dz}{2\pi i} e^{zt} \frac{\hbar}{i} \tilde{F}_{ij}(z),
\]
\[
D_{ij}^{(-)}(t) = \int_{C_z} \frac{dz}{2\pi i} [\theta(t)e^{zt} + \theta(-t)e^{-zt}] \frac{-\hbar}{\beta\hbar} \sum_n \frac{z}{n} \left[ \tilde{F}(z) - \tilde{F}(\nu_n) \right]_{ij},
\]
\[
D_{ij}^{(-3)}(t + i\tau) = \frac{1}{\beta\hbar} \sum_{n=-\infty}^{\infty} e^{i\nu_n \tau} \int_{C_z} \frac{dz}{2\pi i} e^{zt} \frac{-\hbar}{z - \nu_n} [\tilde{F}(z) - \tilde{F}(\nu_n)]_{ij},
\]
\[
D_{ij}^{(33)}(-i\tau) = \frac{1}{\beta\hbar} \sum_{n=-\infty}^{\infty} e^{i\nu_n \tau} \hbar \tilde{F}_{ij}(\nu_n),
\]
(6.3)

where the matrix \( \tilde{F}(\nu_n) \) is given in Eq. (4.17). As mentioned at the end of the fourth section, if we set \( z = i(k - \nu^+) \) in the above expressions, we readily obtain the Fourier-Fourier representation of the propagators. From the expression (6.1) it is trivial to construct the Feynman rule. It is emphasized here that the vertex \( V(\partial/\partial J(t)) \) comes from the anharmonic interaction since the linear-linear interactions are fully taken into the above propagators.

As a simple illustration of the Feynman rule, we consider the system-bath Hamiltonian with the anharmonic interaction \( V(q,t) = gQ^3/3! \) where the system is represented by \( (p_0, q_0, m_0, \omega_0) = (P, Q, M, \Omega) \). The Feynman rule for this system is given through the bilinear corrected propagators \( D_{ij}^{(lm)}(t - s) \) and the three-point vertices illustrated in Fig. 5. By this rule we have the following expression for the first-order correlation function of \( Q \):

\[
-2i\text{Im}(QQ(t)) = \sum_{i,j} \int_{t_0}^{\infty} dt' D_{ij}^{(-+)}(t - t') D_{ij}^{(-)}(t'),
\]
(6.4)

where the lines represent not \( D_{ij}^{(lm)} \) but \( D_{ij}^{(lm)} \). \( (X)_g \) is the connected part of the average of \( X \) by the initial density matrix for the full Hamiltonian including the interacting term \( gQ^3 \). The time evolution on the left-hand side of (6.4) is done also by the full Hamiltonian. Explicitly the leading correction term becomes

\[
\left( \frac{-ig}{\hbar} \right)^2 \int_0^\infty dt' \int_0^\infty dt'' D_{00}^{(--)}(t - t') D_{00}^{(--)}(t' - t'') D_{00}^{(--)}(t'') = \left( \frac{-ig}{\hbar} \right)^2 \int_{C_z} \frac{dz}{2\pi i} e^{zt} D_{00}^{(--)}(z) \Sigma_{00}^{(-)}(z) D_{00}^{(--)}(z),
\]
(6.5)

where

\[
\Sigma_{ij}(z) = \int_0^\infty dt e^{-zt} D_{ij}^{(-)}(t) D_{ij}^{(--)}(t).
\]
(6.6)

We give one more example. Assume we have a nonbilinear coupling \( gQ \sum_i q_i^2/2 \) in addition to the bilinear coupling \( Q \sum_i c_i q_i \). In this case we have only to interpret the lines in Eq. (6.4) slightly differently. In the previous case, the lines always correspond to \( D_{00}^{(lm)} \), while in this case of nonlinear system-bath coupling, the lines imply various \( D_{ij}^{(lm)} \). Namely, the second graph in (6.4) in this case stands for several algebraic expressions. One of them is given by

\[
\left( \frac{-ig}{\hbar} \right)^2 \int_{C_z} \frac{dz}{2\pi i} \sum_{ij} D_{00}^{(--)}(z) \Sigma_{ij}(z) D_{00}^{(--)}(z).
\]

The estimation of these formal expressions together with other quantities in the context of quantum optics needs

\[
\begin{array}{cccc}
- & + & - & + \\
- & + & + & 3 \\
- & + & 3 & 3 \\
\end{array}
\]

FIG. 5. The three-point vertices appearing in the Feynman rule for the system-bath Hamiltonian.
ACKNOWLEDGMENTS

One of the authors (K.O.) is thankful to R. Fukuda for a critical reading of the manuscript.

APPENDIX A: THE PATH INTEGRAL ON THE UNIFIED TIME PATH

In this appendix we introduce the concept of the path integration on the unified time path. We assume \( m = \hbar = 1 \) and consider, as a typical example,

\[
(q'|K_C|q) \equiv \langle q'| T_C e^{-i \int_{t_0}^{t_1} dp^2/2 + \omega^2 p^2/2 - J(t)p} | q \rangle.
\]  

(A1)

As mentioned in Sec. II, we divide the unified time path into \( N+1 \) pieces (\( t_0 = 0, t_1, \ldots, t_N, t_{N+1} = -i\beta \hbar \)) and introduce \( \epsilon_k; \epsilon_i \equiv (\epsilon_0, \epsilon_1, \ldots, \epsilon_N) \) such that

\[
t_i = \sum_{k=0}^{i-1} \epsilon_k, \quad t_i - t_j = \sum_{k=j}^{i-1} \epsilon_k \quad (i > j).
\]  

(A2)

Since we aim at deriving the expression in the limit of large \( N \), we freely deal with \( \epsilon_i \) as if it were an infinitesimal quantity and \( N \) as infinity in the following. Then we obtain the following self-evident generalization of the familiar expression after inserting complete sets of \( q \) and \( p \):

\[
(q'|K_C|q) = \prod_{i=0}^{N} \int \frac{dq_i}{2\pi} \frac{dp_i}{2\pi} e^{iA_q},
\]  

(A3)

where

\[
\begin{pmatrix}
  a_{j-1} + a_j & -b_j & 0 & 0 & \ldots & 0 \\
  -b_j & a_j + a_{j+1} & -b_{j+1} & 0 & \ldots & 0 \\
  0 & -b_{j+1} & a_{j+1} + a_{j+2} & -b_{j+2} & \ldots & 0 \\
  \vdots & \vdots & \vdots & \ddots & \ddots & \vdots \\
  0 & 0 & 0 & \ldots & -b_{k-2} & a_{k-2} + a_{k-1} - b_{k-1} \\
  0 & 0 & 0 & \ldots & 0 & -b_{k-1} + a_{k-1} + a_k
\end{pmatrix}
\]  

(M(1,N))

with \( k > j \) and we have assumed \( J_1 = J_N = 0 \) without loss of generality. Thus, after changing the variable of integration from \( q \) to \( x = q - [M(1,N)]^{-1}d \), we can easily perform the integration:

\[
A_{pq} = \sum_{i=0}^{N} \epsilon_i \left[ \frac{p_i(q_{i+1} - q_i)}{2} - \frac{\epsilon_i p_i^2}{2} - \omega^2 q_i^2 + \epsilon_i J_i q_i \right].
\]  

(A4)

Here we applied the midpoint difference prescription, that is, \( 2q_i = q_{i+1} + q_i \). The integration over \( p_i \) leads to

\[
\langle q'|K_C|q \rangle = \prod_{i=0}^{N} \int \frac{dq_i}{(2\pi\epsilon_i \hbar)^{1/2}} e^{iA_q} \equiv \int [q] e^{i \int_C dt L(t)}.
\]  

(A5)

where

\[
A_q = \int_C dt L(t)
\]

\[
= \sum_{i=0}^{N} \left[ \frac{1}{2\epsilon_i} (q_{i+1} - q_i)^2 - \epsilon_i \omega^2 (q_{i+1} + q_i)^2 + \epsilon_i J_i q_i \right].
\]  

(A6)

If we introduce

\[
a_i = \frac{1}{2\epsilon_i} - \frac{\omega^2}{8} \epsilon_i, \quad b_i = \frac{1}{2\epsilon_i} + \frac{\omega^2}{8} \epsilon_i,
\]  

(A7)

and the vectors

\[
q = (q_1, \ldots, q_N),
\]

\[
d = \left( b_0 q_0, \frac{\epsilon_2}{2} J_2, \frac{\epsilon_3}{2} J_3, \ldots, \frac{\epsilon_{N-1}}{2} J_{N-1}, b_N q_{N+1} \right)
\]  

(A8)

\[
A_q \text{ is expressed as }
\]

\[
A_q = a_0 q_0^2 + a_N q_{N+1}^2 + q \cdot M(1,N) q - 2d \cdot q,
\]  

(A9)

where the \( N \times N \) matrix \( M(1,N) \) is given through

\begin{align*}
M(j,k) & = \begin{pmatrix}
  a_{j-1} + a_j & -b_j & 0 & 0 & \ldots & 0 \\
  -b_j & a_j + a_{j+1} & -b_{j+1} & 0 & \ldots & 0 \\
  0 & -b_{j+1} & a_{j+1} + a_{j+2} & -b_{j+2} & \ldots & 0 \\
  \vdots & \vdots & \vdots & \ddots & \ddots & \vdots \\
  0 & 0 & 0 & \ldots & -b_{k-2} + a_{k-2} + a_{k-1} - b_{k-1} \\
  0 & 0 & 0 & \ldots & 0 & -b_{k-1} + a_{k-1} + a_k
\end{pmatrix},
\end{align*}

(A10)

matrix \( M(1,N) \) as \( D(1,N) \). From now on, we slightly extend the known method of evaluating the inverse of \( M(1,N) \) [28]. Let us consider the determinant \( D(j,k) \) of the matrix \( M(j,k) \). It satisfies the following recurrence relation easily obtained by expansion in terms of the cofactor:

\[
D(j,k) = (a_{k-1} + a_k) D(j,k-1) - b_{k-1}^2 D(j,k-2).
\]  

(A12)

If we introduce
\[ \alpha_k = \frac{1}{2\epsilon_k} - \frac{\omega^2}{8} \epsilon_k + \frac{\omega}{2}, \quad \beta_k = \frac{1}{2\epsilon_k} - \frac{\omega^2}{8} \epsilon_k - \frac{\omega}{2}, \]  

(A13)

we can rewrite the above recurrence formula as

\[ D(j, k) - \alpha_k D(j, k - 1) = \beta_{k-1} [D(j, k - 1) - \alpha_{k-1} D(j, k - 2)] \quad (\alpha \leftrightarrow \beta). \]  

(A14)

It is easy to check that the general solution to this equation is given by

\[ D(j, k) = A \prod_{l=j}^{k} \alpha_l + B \prod_{l=j}^{k} \beta_l \quad (k > j). \]  

(A15)

Noting the conditions

\[ D(j, j) = a_{j-1} + a_j, \quad D(j, j - 1) = 1, \]  

(A16)

we have \( A = \alpha_{j-1}/i\omega, \) \( B = -\beta_{j-1}/i\omega, \) which leads to, in the large \( N \) limit,

\[ D(j, k) = \frac{2}{\omega} \left[ \prod_{l=j-1}^{k} \frac{1}{2\epsilon_l} \right] \sin \omega(t_{k+1} - t_{j+1}) \quad (k > j). \]  

(A17)

Thus we have a formula,

\[ 2\epsilon_{j-1} \tilde{D}(j, k) = \frac{2 \sin \omega(t_k - t_j)}{\omega} \quad (k > j), \]  

(A18)

where

\[ \tilde{D}(j, k) = D(j, k) \prod_{l=j}^{k} 2\epsilon_l. \]  

(A19)

Thereby the prefactor in (A11) is given by

\[ \left[ \prod_{l=0}^{N} (2\pi\epsilon_l i)^{-1/2} \right] \left[ \frac{(\pi i)^{N}}{D(1, N)} \right]^{1/2} = \frac{1}{2\pi \epsilon_0 i \tilde{D}(1, N)} \]  

\[ = \frac{\omega}{2\pi \sinh \omega \beta}, \]  

(A20)

Since the \( i-j \) element of the cofactor of the matrix \( M \) is given by, for \( j > i, \)

\[ \Delta_{ij} = D(1, i - 1) \left( \prod_{l=i}^{j-1} b_l \right) D(j + 1, N) \]  

\[ \approx \frac{2\epsilon_j}{\prod_{l=1}^{N} 2\epsilon_l} D(1, i - 1) \tilde{D}(j + 1, N), \]  

(A21)

we get, for \( j > i, \)

\[ [M(1, N)]_{ji}^{-1} = \frac{\Delta_{ij}}{D(1, N)} \]  

\[ = \frac{\tilde{D}(1, i - 1)}{\tilde{D}(1, N)} 2\epsilon_j \tilde{D}(j + 1, N). \]  

(A22)

Thus we have

\[ [M(1, N)]_{ji}^{-1} = \theta(j - i) \frac{2 \sin \omega(t_i - t_j) \sin \omega(t_j - t_N)}{\omega \sin(-i\omega \beta)} + (i \leftrightarrow j). \]  

(A23)

By recovering \( m \) and \( \hbar \) we arrive at the result given in (2.19).

**APPENDIX B: DERIVATION OF EQ. (2.26)**

Equation (2.26) is simply derived by using the relation

\[ \int dx_i \cdots dx_n e^{-\frac{1}{2} x_i (A + B) x_i + J_i x_i} \]  

\[ = e^{-\frac{1}{2} \lambda_i A_i \lambda_j B_{ij}} \int dx_i \cdots dx_n e^{-\frac{1}{2} x_i B_{ij} x_j + J_i x_i} \]  

(B1)

and the well-known identity

\[ \int dx_i \cdots dx_n e^{-\frac{1}{2} x_i A_i x_j + J_i x_i} = (2\pi)^{n/2} (\det A)^{1/2} e^{\frac{1}{2} J_j A_{ij}^{-1} J_i}. \]  

(B2)

[24] H. Grabert, P. Schramm, and G.-L. Ingold, Phys. Rep. 168, 115 (1988). Though the notation is different, their $\tilde{F}(q, q', \tilde{q})$ is essentially the same quantity as our $Z_{14}^{(0)}$.