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Scaling Crossover in Crack-Tip Stresses and a Robust Scaling Law for Fracture Strength

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We study the crack-tip stress in a two-dimensional network model with a practical nonlinear stress–strain relation. As a result, we find a scaling crossover in the relation between the crack-tip stress and mesh size from a linear to nonlinear scaling regime, leading to a simple scaling law for the fracture strength of the network. The present results may be pertinent to the strength of materials with voids. In this context, the results are independent of the detailed geometry of voids, and the scaling law for the strength suggests that nonlinearity in a high stress–strain region is essential for material strength.

1. Introduction

Materials start failing at a crack tip because stress is concentrated at the tip.[1,2] The control of stress concentration is a key to enhance the toughness of materials, as seen in many tough biological materials.[3–5] For example, in the layered structure of nacre,[6,7] and in the spiral structure of a crustacean's exoskeleton,[8–10] stress concentrations are reduced by virtue of their remarkable structures.[11] In spider webs, which represent a natural lightweight tough structure,[12] stress concentrations are absent.[13]

The inclusion of voids may be another strategy employed by nature to strengthen biological materials. In the stereom in adult skeletal plates of echinoderms or holothurians (e.g., sea cucumbers) many voids in the structure[14] may contribute to the resilient mechanical response. Voids in apples may also play an important role in their toughness.[15] The void structure of the skeleton of biosilica in certain sponges may also reinforce the structure.[16] In addition, recent studies have revealed that many biological materials, for example, spider webs[12] and nacre,[17] exhibit nonlinearity in their stress–strain relation.

Although there have been many studies on the stress concentration in materials with voids, scaling laws for fracture mechanical properties for *nonlinear materials with voids* have yet to be explored. (Specifically, a linear or nonlinear material here refers to a material for which the stress–strain relation is linear or nonlinear, respectively.) For *linear materials*, simple relations between failure stress and void size have been suggested, for example, from the results of an experiment,[18] and a scaling law between the maximum force and void size has been shown using a lattice model.[19,20] For *cellular solids*, a number of scaling laws have been proposed,[21] on the basis of beam theory considering the moment of forces. (A cellular solid is a material composed of small cells, such as solidified polymer foams and materials with a honeycomb structure, where each cell can be closed or opened, i.e., each cell is composed of ridges and faces or only of ridges.) However, most of the results are limited to cases in which the cell structure can be characterized by the width and length of edges and the thickness of faces. In addition, most of the studies on cellular solids are based on linear fracture mechanics except for a few examples.[22] For *nonlinear materials* without voids, the crack tip singularity has been analyzed[23,24] using a nonlinear model that can describe elastoplastic materials with the understanding that no unloading occurs. For *nonlinear materials with voids*, local processes such as nucleation, growth, and coalescence, and detailed analysis or numerical behaviors have been actively studied.[25–28] In addition, a relation between crack-tip stress and size of voids was proposed.[29] However, the nonlinear stress–strain relation employed in the study is applicable only to a limited class of materials.

In this study, we employ another nonlinear stress–strain relation that has been used to describe many realistic materials.[23,24] As a result, we find a scaling crossover in the relation between crack-tip stress and void size. In this article, this crossover is first conjectured on the basis of scaling arguments independent of the detailed geometry of voids, and then confirmed by a simple network model. The crossover allows us to confirm that a failure stress formula derived for a simpler nonlinear model is also valid in more practical nonlinear models with an appropriate interpretation, highlighting a number of design principles that might be useful for developing artificial materials with voids.

2. Conjectures Based on Nonlinearly Extended Griffith Theory

We consider the classic Ramberg–Osgood model for work hardening, where the following two different relations between stress (σ) and strain (ε) match each other at a crossover strain ε_c as shown in Fig. 1(b):

$$\sigma = \begin{cases} E\varepsilon & \varepsilon < \varepsilon_c \\ \alpha E \varepsilon^{1/n} & \varepsilon > \varepsilon_c \end{cases}, \qquad (1)$$

where E is the linear elastic modulus, αE is the nonlinear modulus ($\alpha = \varepsilon_c^{1-1/n}$), and n is a positive number larger than one.

2.1 *The case of $\varepsilon_c = 0$*

In this simple nonlinear case, Griffith's well-known expression for failure stress was generalized into a nonlinear form[29] that has been confirmed experimentally.[30] Griffith's failure stress is given as $\sigma_F \sim \sqrt{E\gamma/a}$ at the level of scaling laws (i.e., setting the numerical coefficient to one for simplicity), where γ is the fracture energy (per area) and a is the size of a line crack: The failure stress becomes smaller as the crack size increases. The nonlinear generalized version is given as

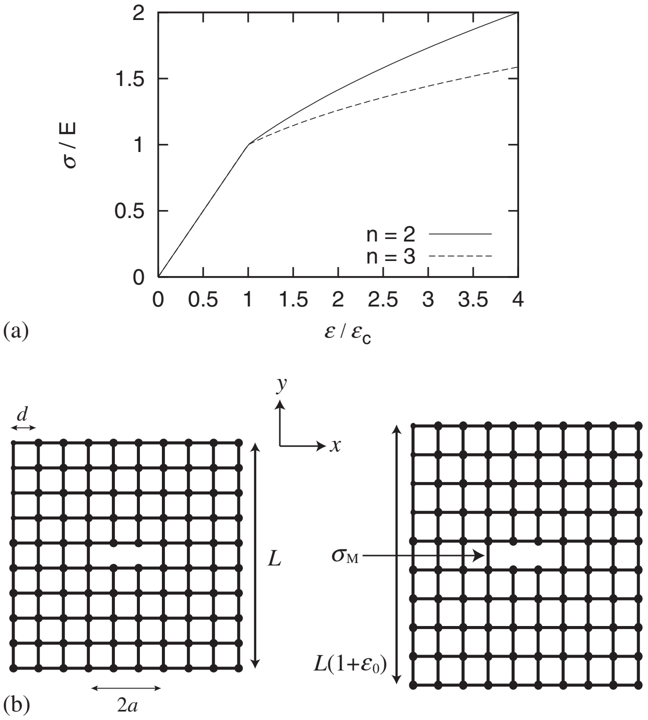

Fig. 1. (a) Stress σ normalized by the modulus E vs strain normalized by the crossover strain ε_c, employed in the present calculations. (b) Schematic illustration of the meshwork with a crack of half-size a for numerical calculations before (left) and after (right) a stretch with strain ε_0. The initial system size is L, the mesh size is d, and the maximum stress in the system is σ_M.

$$\sigma_F \sim \alpha E\left(\frac{\gamma}{\alpha E a}\right)^{1/(n+1)}. \qquad (2)$$

Note that this relation reduces to Griffith's failure formula when $n = 1$ and $\alpha = 1$.

At the level of scaling laws, this nonlinear formula can be justified as follows. We introduce a line crack of size a in an infinitely large (thick or thin) plate subject to the plane strain or stress conditions and consider the energy balance at an equilibrium: the elastic energy released by virtue of the existence of the crack is balanced with the energy required to create fracture surfaces. Dimensionally, the elastic energy is estimated as $\sigma\varepsilon a^2$ per unit thickness of the sample because the characteristic elastic energy per unit volume scales as $\sigma\varepsilon$, whereas the fracture energy is estimated as γa per unit thickness. In the balance, $\sigma\varepsilon a^2 \sim \gamma a$, the stress σ satisfies the second expression in Eq. (1) because the first expression is never satisfied when $\varepsilon_c = 0$. In this way, we obtain the failure stress σ_F in Eq. (2).

Equation (2) has been shown to be physically equivalent to the well-known singularity at the crack tip for the present nonlinear model:[23,24]

$$\sigma(r) \sim \sigma_0(a/r)^{1/(n+1)} \quad \text{when } r \ll a. \qquad (3)$$

Here, r is the distance from one of the crack tips and σ_0 is the remote stress.

From Eq. (3), we can conjecture a formula for the crack-tip stress in materials with voids. The divergence of $\sigma(r)$ in the limit $r \to 0$ is never attained in real materials. If the sample possesses a void structure characterized by the size d (which is larger than the characteristic sizes of practically unremovable defects), the continuum description breaks down at the scale of d, i.e., Eq. (3) no longer holds below $r \sim d$ (if there are a number of length scales characterizing the void structure, the cutoff scale d is identified with the largest scale). The cutoff of the singularity at the scale of d might imply that the stress field in the system has the maximum σ_M that scales with the cutoff value of stress $\sigma(r = d)$ as $\sigma_M \sim \sigma_0(a/d)^{1/(n+1)}$. This conjecture has been confirmed in a simple network model.[29]

2.2 The case of $\varepsilon_c \neq 0$

In the above derivation of σ_M in the case of $\varepsilon_c = 0$, we assumed that the stress σ satisfies the second nonlinear expression in Eq. (1). However, when $\varepsilon_c \neq 0$, the mechanical behavior can be completely linear when ε is small. In such a case, the relations $\sigma = E\varepsilon$ and $\sigma_M \sim \sigma_0(a/d)^{1/2}$ should be satisfied instead. This happens when the maximum stress set by the remote stress σ_0 in the linear regime is smaller than the crossover stress: $\sigma_0(a/d)^{1/2} < E\varepsilon_c$. This condition for the linear regime is cast in the following form:

$$\varepsilon_0 \equiv \sigma_0/E < \varepsilon_c(d/a)^{1/2}. \qquad (4)$$

In contrast, when the nonlinear version $\sigma_M \sim \sigma_0(a/d)^{1/(n+1)}$ is valid, the condition $\varepsilon_0 > \varepsilon_c$ should be satisfied because, in this case, the nonlinear stress–strain relation is valid near the crack [i.e., in the region $r \lesssim a$ in Eq. (3)].

In summary, when $\varepsilon_c \neq 0$, we conjecture, from the above naive scaling arguments, the following scaling crossover for $a \gg d$ (where k_1 and k_2 are numerical coefficients):

$$\sigma_M = \begin{cases} k_1\sigma_0(a/d)^{1/2} & \varepsilon_0 < \varepsilon_c(d/a)^{1/2} \\ k_2\sigma_0(a/d)^{1/(n+1)} & \varepsilon_0 > \varepsilon_c \end{cases}. \qquad (5)$$

For later convenience, we further rewrite Eqs. (1) and (5) in the following forms:

$$\tilde{\sigma} = \begin{cases} \tilde{\varepsilon} & \text{for } \tilde{\varepsilon} < 1 \\ \tilde{\varepsilon}^{1/n} & \text{for } \tilde{\varepsilon} > 1 \end{cases}, \qquad (6)$$

$$\tilde{\sigma}_M = \begin{cases} k_1(a/d)^{1/2} & \text{for } \tilde{\varepsilon}_0 < (d/a)^{1/2} \\ k_2(a/d)^{1/(n+1)} & \text{for } \tilde{\varepsilon}_0 > 1 \end{cases}. \qquad (7)$$

Here, we have introduced the renormalized variables $\tilde{\sigma} = \sigma/E\varepsilon_c$, $\tilde{\sigma}_M = \sigma_M/\sigma_0$, $\tilde{\varepsilon} = \varepsilon/\varepsilon_c$, and $\tilde{\varepsilon}_0 = \varepsilon_0/\varepsilon_c$. In these representations, the forms are universal in the sense that they are independent of ε_c.

The relations derived above are universal in a different sense in that they are independent of the geometry of the voids. This is because, in the derivation, we tacitly assumed that the modulus E is independent of d when introducing ε_0 in Eq. (4), which can be justified only by comparing systems with the same E value. In general, the elastic modulus E is dependent on the geometry of voids in a complex manner: Even if the modulus E is expressed, for example, simply as $E = \phi E_s$ (ϕ is the volume fraction of the material and E_s is the modulus of the material without voids), ϕ depends on the detailed geometry of the voids. This implies that if we deal with systems with different E values, the relations will become geometry-dependent, containing extra d-dependences originating from the d-dependence of E in the definition of ε_0 in Eq. (4). In other words, even though the above relations are valid only for comparing systems with the same E value, the relations are universal or geometry independent. For further understanding of this point, the paragraph below Eq. (8) and the Appendix should be helpful.

To confirm such geometry-independent relations in numerical calculations, we use the simplest model on the basis of the following principle. Since the relations are conjectured through geometry-dependent arguments, if we can confirm relations in a specific geometry, we can expect that the relations will be valid irrespective of the detailed geometry.

In other words, we consider that materials with voids can be characterized by a length scale corresponding to the typical size of voids, d, and that the simplest minimum model describing such systems will be the lattice model of lattice size d. This is because we can draw the above conjecture (proved below by the lattice model) simply by using a very general property in a geometrically independent manner: the material in question has a certain length scale below which the continuum description fails. Accordingly, if we can show the conjecture in a specific (and the simplest) geometry, we can expect that the scaling structure will be the same for the lattice model and for materials with voids whose largest length scale is d.

3. Simulation Model

In order to confirm the crossover of the scaling exponent from the linear value $1/2$ to the nonlinear value $1/(n+1)$ suggested in Eq. (5), we perform numerical calculations for a two-dimensional meshwork with mesh size d [Fig. 1(a)]. This system is composed of nodal points. Each nearest-neighbor pair is connected with a spring of length d as in Ref. 29 but the loading mechanical property of the springs is governed by Eq. (1). Note that we consider below only stretched equilibrium states so that no unloading curves are necessary. In the calculations, we keep the bulk elasticity fixed while changing the mesh size following the assumption of a fixed E. The trick useful for this is explained in the Appendix.

We introduce a line crack of size a propagating in the x-direction in the nodal system by removing springs located at the crack [Fig. 1(a)]. We then stretch the system at both edges in the y-direction with a fixed strain of magnitude ε_0 and obtain the equilibrium state by a relaxation method.[29] As expected, the maximum force f_M appears in the spring located at the crack tip [see Fig. 1(a)]; the maximum stress is defined as $\sigma_M = f_M/d$.

For numerical convenience, we fixed the crack size a to $40d_0$ (for the condition $a \gg d$ to be satisfied) and changed the mesh size d from d_0 to $2d_0$ to $4d_0$, whereas the system size L was fixed to $400d_0 \times 400d_0$ (for the condition $L \gg a$ to be satisfied). Here, d_0 is the unit length used for the numerical calculations. In fact, physically, it is not compulsory to introduce the quantity d_0. It is physically equivalent to have fixed the system size L and introduced a crack of size $a = L/10$, and made calculations for three different values of mesh size d (i.e., $L/400$, $L/200$, and $L/100$), in order to investigate the effect of a change in the void size on the maximum stress appearing in the system. (At the same time, we can consider that we have changed the ratio a/d while fixing d, with L always much larger than a).

4. Results

The crossover of the scaling regimes for the maximum stress is confirmed in Fig. 2(a), for $n = 2$ and $\varepsilon_c = 1$, where

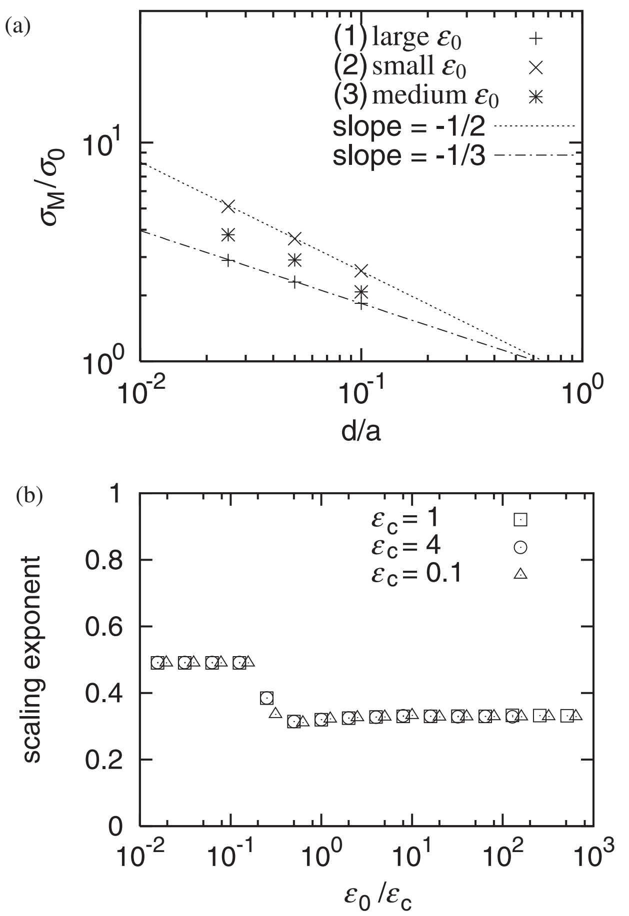

Fig. 2. (a) Normalized maximum stress σ_M/σ_0 as a function of normalized mesh size d/a for three values of remote strain ε_0: (1) 2^6, (2) 2^{-6}, and (3) 2^{-2}. Here, the crossover strain ε_c is 1 and the nonlinear exponent n is 2. The slopes $-1/2$ and $-1/3$ correspond to the predictions of Eq. (5). (b) Crossover of the scaling exponent for the maximum stress σ_M/σ_0 as a function of the remote strain $\varepsilon_0/\varepsilon_c$. The crossover is expressed as the universal curve for three different values of crossover strain ε_c.

we plot the normalized maximum stress σ_M/σ_0 as a function of the normalized mesh size d/a for three values of the remote strain ε_0. When the value of ε_0 is large, as in Fig. 2(a1), the calculated data are close to a straight line and the negative of the slope of the line corresponding to the scaling exponent is close to $1/(n+1) = 1/3$, as predicted in Eq. (5), with $k_1 = 0.817$. When the value of ε_0 is small, as in Fig. 2(a2), the calculated data are again close to a straight line and the exponent obtained in the same way is close to $1/2$, as predicted, with $k_2 = 0.867$. When the value of ε_0 is between the values of those in (1) and (2), the data are not perfectly on a straight line, as in Fig. 2(a3); the slope defined by the first two points are smaller than the slope defined by the second and third points. This slight difference in the slope can be understood as follows. When d/a is smaller (the first two points), the condition for the linear regime $\varepsilon_0 < \varepsilon_c(d/a)^{1/2}$ is less satisfied, which explains why the slope defined by the first two points is smaller, i.e., closer to the nonlinear regime, than the slope defined by the second and third points in Fig. 2(a3).

The universal feature, or the independence of the results from ε_c, seen in Eqs. (6) and (7), is confirmed in Fig. 2(b). In the figure, the crossover of the scaling regime for the maximum stress is demonstrated as a function of the renormalized remote strain ε_0. We see here that the crossover curves collapse onto a single master curve. The scaling exponent takes the linear value of $1/2$ when $\tilde{\varepsilon}_0 < (d/a)^{1/2}$, whereas it approaches the nonlinear value of $1/3$ when $\tilde{\varepsilon}_0 > 1$; between the two regimes, there is a transition region. Note that $(d/a)^{1/2}$ is approximately 0.1 because we determine the slope from the first two points ($d/a = 1/40$ and $2/40$) in the plots, as we did in Fig. 2(a). This explains why the transition region starts at approximately $\tilde{\varepsilon}_0 \sim 0.1$. We have also verified the crossover for $n = 3$ through numerical calculations, with the exponent changed from $1/2$ to $1/4$.

Equation (7) thus confirmed suggests that even if the strain is not so large and the tip stress is governed by the linear scaling law (or by the nonlinear law when the strain is below the failure threshold), the alleviation of the concentrated crack-tip stress by the cutoff scale d is advantageous in terms of fatigue tolerance.

5. Discussion

Here we introduce an inherent material strength, σ_s, in such a way that the material starts to break when the local stress exceeds σ_s. This stress is a material constant on a scale much smaller than the void size d. In other words, σ_s is the inherent material strength of a sample without voids of the same material. In view of the nonlinear Griffith formula, Eq. (2), this critical stress σ_s can be identified with the failure stress σ_F of the material in the absence of macroscopic cracks, namely, $\sigma_s \sim E(\gamma/Ea_0)^{1/(1+n)}$ where a_0 ($\ll d$) is the size of Griffith flaws, i.e., the size of defects which act as a small crack in a system without macroscopic cracks. Note here that, as shown explicitly in the Appendix, in our numerical model where the bulk elasticity is fixed, the stress σ_s is the same for different d values.

The introduction of the inherent material strength σ_s allows us to obtain a formula for strength. The assumption of the stress–strain relation in Eq. (1) tacitly requires the condition $\sigma_s > E\varepsilon_c$ (otherwise the material is simply linear). Thus, the crack tip stress at the moment of failure is always given by the nonlinear expression $\sigma_M \sim \sigma_0(a/d)^{1/(n+1)}$. This, with the failure condition (set by the definition of σ_s) $\sigma_M \sim \sigma_s$, provides the following expression for the failure strength:

$$\sigma_F \sim (d/a)^{1/(n+1)}\sigma_s \quad \text{for } a \gg d. \tag{8}$$

We stress here that Eq. (1) is robust in the following two aspects. (I) As seen from the above derivation, *this formula is valid for almost any nonlinear material only if the stress–strain relation near $\sigma = \sigma_s$ is approximately described by the relation $\sigma \sim \varepsilon^{1/n}$*. In fact, this formula was derived and confirmed using a simpler nonlinear model ($\varepsilon_c = 0$),[29,31] but here we find that the same formula is obtained for the present more practical nonlinear model. In addition, in Eq. (1), we consider a piecewise nonlinear stress–strain relation composed of two scaling regimes, but Eq. (5) is not limited to this case; even if a piecewise relation is composed of several scaling regimes, as has been discussed for spider silk,[12] Eq. (5) is valid only if the final scaling regime near $\sigma = \sigma_s$ is described by $\sigma \sim \varepsilon^{1/n}$. In other words, *high nonlinearity in a large stress–strain region is very important for achieving a high material strength*. (II) *This formula is also valid irrespective of the geometry of voids when comparing materials with the same E value* [for the reason described in the Appendix]. In other words, if we change the void size while keeping the porosity $m \cdot 4\pi r^3/3$ at a fixed value (m is the number of voids per unit volume of the material), then, although a series of materials with different r (but with the same porosity) have the same bulk elasticity, the ones with larger voids are stronger. In fact, Eq. (8) is compatible with previous results for open-cell and closed-cell solids,[21] and this equation extends the previous results to nonlinear cases in which voids are not necessarily described by open or closed cells: if, in their formulas, we neglect the d-dependence originating from the d-dependence of the bulk elastic constant, their formulas reduce to the linear version of Eq. (8).

The scaling law in Eq. (8) for the failure stress σ_F implies the following design principles for strong materials with voids: it is advantageous to make the void scale d and the nonlinear index n larger if the bulk elastic modulus is fixed to a constant value. Note that the optimized size for the mesh size d should be determined from some other practical requirements for the material. For example, if the material needs to be transparent, d should be slightly smaller than the wavelength of light in question.

The strength discussed here is for samples with macroscopic cracks (not for samples without macroscopic cracks whose strength is dictated by small flaws or defects) when the samples possess a largest characteristic void size that is much smaller than the macroscopic cracks. In other words, we consider that the development of macrocracks is practically unavoidable in a material and further consider that under what conditions we can use the material with macroscopic cracks safely without leading to failure, as usually done in fracture mechanics. In such a case, the cutoff of the crack-tip stress singularity occurs on the largest characteristic scale of the voids, and our suggestions are not undermined by the existence of nanometer-size flaws, which are not removable in practice.

We expect that our result will be applicable to real ductile materials with voids to some extent, although there is a natural limitation because the effect of energy dissipation is neglected in our demonstration. Here, we point out that the nonlinear stress–strain relation employed above is frequently mentioned as the simplest model for plastic materials. When the index n is large enough, stress increases linearly with strain to a certain point above which stress becomes constant, and this constant value can be regarded as a yield stress. In this sense, our result may provide a useful starting point even when we consider the strength of ductile materials with voids.

The design principles confirmed here in a rather robust way might be useful, to some extent, even for plastic nets used in anticlogging, buildings, bridges, and space structures, especially because our results are independent of the detailed geometry of voids and applicable to many practical nonlinear models. The principles might suggest how the combination of voids and nonlinear elastic property found in nature provides high strength. Connections with biological materials, however, should be explored in the future, which may require the involvement of biologists.

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Appendix

The inherent stress σ_s should be generically unchanged when the void size is varied with the bulk elasticity fixed even in the network model. We first explain this for the two-dimensional (2D) case and then for the three-dimensional (3D) case.

A.1 Two-dimensional case

We first consider a specific example, in which a 2D system of mesh size d is composed of nonlinear springs of natural length d that break when the applied force on them exceeds f_s. Then, the failure occurs when the maximum force $f_M = \sigma_M d$ with $\sigma_M \sim \sigma_0 (a/d)^{1/(n+1)}$ matches f_s. By comparing this balancing equation with Eq. (8), we see that σ_s scales as f_s/d because σ_0 in the balancing equation is identified with σ_F.

We now multiply the mesh size by m to obtain a system with a larger mesh size md. The new system is composed of springs of natural length md. To retain the same bulk elasticity, each spring of length md should be a composite spring comprising m^2 original springs of length d, i.e., m serial connections of m parallel connections of springs of length d. In this way, the original system with mesh size d and the new system with mesh size md are composed of the same number of original springs of natural length d. Thus, the two systems are expected to have the same bulk elasticity. This is mathematically shown to be true in Ref. 29 (and is explicitly shown below for the 3D case).

For the composite springs of length md, each of which is m serial connections of m parallel connections of the original spring of length d, the critical force for failure should be generically given simply by mf_s. Then, the failure occurs when the maximum force $\sigma_M md$ with $\sigma_M \sim \sigma_0 (a/(md))^{1/(n+1)}$ matches mf_c. This means again that σ_s scales as f_s/d, which is independent of m. These arguments justify the naive assumption.

A.2 Three-dimensional case

The above arguments can be generalized for the 3D system of mesh size d. To understand this, we first check that, even in the 3D case, the same bulk elasticity can be kept when the mesh size is increased to md by using the same number of original springs of length d. Here, the original spring is a nonlinear spring following the relation, $F = k\Delta x^{1/n}$, where F is the applied force at the ends and Δx is the elongation. Since the number of nodal points is decreased from the original number N^3 to $(N/m)^3$ by the increase in the mesh size, a composite spring of length md should be a composite of m^3 original springs, i.e., a composite of m serial connections of m^2 parallel connections in the 3D case. When this composite spring is stretched by Δx, the original springs of length d are all stretched by $\Delta x/m$ so that the force F_c at the ends of the composite spring is given by $F_c = m^2 k(\Delta x/m)^{1/n}$. The stress defined by $\sigma \equiv F_c/(md)^2$ should be expressed as $\sigma = \mu \varepsilon^{1/n}$ with the strain given by $\varepsilon = \Delta x/(md)$. Then, a simple mathematical manipulation leads to an m-independent elastic modulus, $\mu = kd^{1/n-2}$. This justifies that the same bulk elasticity is maintained upon increasing the mesh size by using the same number of original springs. Then, we readily understand in the 3D case that σ_s scales as f_s/d^2, which is again independent of m [e.g., $f_M = \sigma_M d$ or $\sigma_M md$ in the 2D case is replaced with $f_M = \sigma_M d^2$ or $\sigma_M (md)^2$]. In this way, Eq. (8) with constant σ_s for materials with the same bulk elasticity is also valid in the 3D case.

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